Inönü–Wigner contractions of the real four-dimensional Lie algebras

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All Inönü–Wigner contractions of the real four-dimensional Lie algebras are found. The results are summarized in tables.

1. INTRODUCTION

In recent years, the low-dimensional real Lie algebras have come under intensive study. A complete classification into isomorphism classes of all the real Lie algebras of dimension less than or equal to five has been found.[1] The real nilpotent Lie algebras of dimension six have also been classified into isomorphism classes.[2] A complete classification into conjugacy classes of all the subalgebras of the real Lie algebras of dimension less than or equal to four has been found.[3] All Casimir invariants, all rational invariants, and all general invariants of all real Lie algebras of dimension less than or equal to five and of all real nilpotent Lie algebras of dimension six have been calculated.[4] All Inönü–Wigner contractions of all real Lie algebras of dimension less than or equal to three have been found.[5] The deformations of the three-dimensional real Lie algebras have been studied.[6]

This interest in low-dimensional réal Lie algebras (groups) stems mainly from the fact that the low-dimensional Lie algebras (groups) occur as subalgebras (subgroups) of higher-dimensional Lie algebras (groups) that are likely to be of direct concern in physical applications. For example, in the theory of induced representations of groups, representations of subgroups are used to construct representations of the full group.[7] In representation theory, chains of subgroups of a group can provide sets of commuting operators whose eigenfunctions provide bases of representation spaces for the group. Knowledge of the subgroups of a symmetry group supplies an approach to the study of broken symmetries.[8] Furthermore, the low-dimensional Lie algebras (groups) are of interest per se, by providing a convenient supply of examples to use as a basis for trying to extend the mathematical theory of the structure and properties of real Lie algebras (groups) and their representations.

Segal[9] was first to suggest a kind of limiting process among Lie groups and Lie algebras. This was followed by the studies of Inönü and Wigner,[10] where a different limiting process was introduced and called a contraction. Saletan[11] generalized the notion of contraction in such a way that the Inönü–Wigner contraction appeared as a special case. Doebner and Melsheimer[12] introduced another generalized notion of contraction, called a p contraction, also having the Inönü–Wigner contraction as a special case. Conatser[13] has studied some of the relationships among these various ideas of contraction.

Numerous physical applications of contractions of Lie groups and Lie algebras have appeared. In a study of dynamical group descriptions of interacting systems,[14] the contraction limit corresponded to the coupling constant going to zero giving noninteracting systems. Contractions were used to explore the relationships among various kinematical groups that had been derived and to gain some insight into their physical meaning.[15] The Wigner coefficients of the three-dimensional Euclidean group were obtained by contraction from the Wigner coefficients of the four-dimensional special orthogonal group.[16] The relationship between the conformal group and the Schrödinger group was elucidated using contractions.[17] Contractions were used[18] to study the relationships among various Lie algebras admitting a relativistic position operator.[19] Finally, a contraction of the structural group of a fiber bundle with Cartan connection was studied in a proposed gauge theory of strongly interacting hadrons.[20] It seems that the relatively simple Inönü–Wigner contraction has found the most widespread application.

The purpose of this paper is to find all Inönü–Wigner contractions of all four dimensional real Lie algebras. In Sec. 2, the method of Inönü–Wigner contraction is briefly discussed and a few basic theorems are mentioned. In Sec. 3, the results of the contractions of the four-dimensional real Lie algebras are presented. Section 4 contains some concluding remarks.

2. INÖNÜ–WIGNER CONTRACTIONS

$L=(V,\mu)$ will denote a Lie algebra, where V is the underlying vector space over the field of real numbers and μ is the binary operation from $V\times V$ into V specifying the Lie bracket of any two elements X and Y of V, $\mu(X,Y)=[X,Y]$. Let $\{X_i\}$, $i=1,2,\ldots,n=\dim V$, be a basis for V; then

$$\mu(X_i,X_j)=[X_i,X_j]=\sum_{k=1}^{n} c_{ij}^k X_k,$$

where the numbers c_{ij}^k are the structure constants of the Lie algebra.

Let T_α be a family of linear operators on V, parametrized by α, $0\le\alpha\le 1$, that is nonsingular for $\alpha\ne 0$. Define a family of Lie algebras $L_\alpha=(V,\mu_\alpha)$, where

$$\mu_\alpha(X_i,X_j)=[X_i,X_j]_\alpha=T_\alpha^{-1}[T_\alpha X_i,T_\alpha X_j], \quad (1)$$

for all $i,j=1,\ldots,n$. Suppose T_α depends linearly on α and that there are complementary subspaces W and U of V such that T_α decomposes as

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$$T_\alpha = \begin{pmatrix} I_W & 0 \\ 0 & 0 \end{pmatrix} + \alpha \begin{pmatrix} A & 0 \\ 0 & I_U \end{pmatrix}, \tag{2}$$

where I_W and I_U are the identity operators on W and U, respectively, and A is an operator on W. Then, Inönü and Wigner[10] have shown that $\lim_{\alpha\to 0} L_\alpha^W = L_0^W$ exists if and only if W is a subalgebra of L. L_0^W is called the Inönü–Wigner contraction of L with respect to W. The operator A in (2) can be taken equal to 0 without loss of generality. It was shown[11] that the result of contraction is independent of the choice of the subspace U complementary to the subalgebra W. Furthermore, the result of contraction is independent of the choice of representative subalgebra from a given conjugacy class of subalgebras.

To calculate all Inönü–Wigner contractions of a Lie algebra L:

(1) find all conjugacy classes of subalgebras of L;

(2) select a representative for each conjugacy class;

(3) select a basis $\{X_1, \ldots, X_p\}$ for each representative subalgebra W, where $p = \dim W$;

(4) find a complementary subspace U and select a basis $\{X_{p+1}, \ldots, X_n\}$ for it, where $n = \dim V$;

(5) let T_α act on the basis elements according to $T_\alpha X_i = X_i$ for $i = 1, \ldots, p$, and $T_\alpha X_i = \alpha X_i$ for $i = p+1, \ldots, n$;

(6) calculate the Lie brackets $[X_i, X_j]_\alpha$, for all i and j, using Eq. (1);

(7) take the limit of α going to zero;

(8) identify the resulting contracted algebra L_0^W.

In Sec. 3, the results of applying this prescription to the case $n = 4$ are presented.

A contraction for which $L_0^W = nA_1$, the n-dimensional Abelian Lie algebra, is termed trivial. If $L_0^W = L$, the contraction is called improper. Let L' denote the derived algebra of L. Several theorems are now listed without proofs.[13]

(1) If $W = L$, then $L_0^W = L$.

(2) If $W = \{0\}$, then $L_0^W = nA_1$.

Thus, contractions with respect to improper subalgebras are either improper contractions or trivial contractions.

(3) If $L = nA_1$, then $L_0^W = nA_1$, for all W. Abelian Lie algebras have only trivial contractions.

(4) If $W \subset Z$, the center of L, then $L_0^W = nA_1$.

(5) For any contraction, $\dim L_0^{W\prime} \leqslant \dim L'$.

3. CONTRACTIONS OF THE FOUR-DIMENSIONAL REAL LIE ALGEBRAS

Each table displays the results of Inönü–Wigner contraction for one four-dimensional Lie algebra. In the table, the nonzero Lie brackets of the algebra are given. The representatives of the conjugacy classes of proper subalgebras of the algebra[3] are presented in the first column. All subalgebras leading to the same contracted algebra are listed together. The improper subalgebras are omitted. The contracted algebras are given in the second column. A table for the Abelian algebra $4A_1$ is omitted.

4. CONCLUDING REMARKS

The results of this paper are summarized in Tables I–XXIX. Several observations can be made. Two of the algebras (see Tables III, XX) have only improper and trivial contractions. Three of the algebras (see Tables XXII, XXVII, XXVIII) have no improper contractions with respect to proper subalgebras. There are many examples where the equality in Theorem 5 is obtained.

TABLE I. Contractions of $A_2 \oplus 2A_1$. The range of parameters: $-\infty < x < \infty$, $0 \leqslant \phi < \pi$, $\epsilon = \pm 1$.

Nonzero Lie brackets $[e_1, e_2] = e_2$	
Subalgebra representative	Contracted algebra
$\{e_2\}$, $\{e_3\cos\phi + e_4\sin\phi\}$, $\{e_3, e_4\}$, $\{e_2, e_3\sin\phi - e_4\cos\phi\}$, $\{e_2, e_3, e_4\}$	$4A_1$
$\{e_1 + x(e_3\cos\phi + e_4\sin\phi)\}$, $\{e_1, e_3, e_4\}$, $\{e_1 + x(e_3\cos\phi + e_4\sin\phi), e_2\}$, $\{e_1 + x(e_3\cos\phi + e_4\sin\phi),\ e_3\sin\phi - e_4\cos\phi\}$, $\{e_1 + x(e_3\cos\phi + e_4\sin\phi), e_3\sin\phi - e_4\cos\phi, e_2\}$	$A_2 \oplus 2A_1$
$\{e_2 + \epsilon(e_3\cos\phi + e_4\sin\phi)\}$, $\{e_2 + \epsilon(e_3\cos\phi + e_4\sin\phi), e_3\sin\phi - e_4\cos\phi\}$	$A_{3,1} \oplus A_1$

TABLE II. Contractions of $2A_2$. The range of parameters: $-\infty < x < \infty$, $\epsilon = \pm 1$.

Nonzero Lie brackets $[e_1, e_2] = e_2$, $[e_3, e_4] = e_4$	
Subalgebra representative	Contracted algebra
$\{e_2\}$, $\{e_4\}$, $\{e_2, e_4\}$	$4A_1$
$\{e_1\}$, $\{e_3\}$, $\{e_1, e_2\}$, $\{e_3, e_4\}$, $\{e_1, e_4\}$, $\{e_2, e_3\}$, $\{e_1, e_2, e_4\}$, $\{e_2, e_3, e_4\}$	$A_2 \oplus 2A_1$
$\{e_1, e_3\}$, $\{e_1, e_2, e_3\}$, $\{e_1, e_3, e_4\}$	$2A_2$
$\{e_2 + \epsilon e_4\}$	$A_{3,1} \oplus A_1$
$\{e_1 + e_3\}$, $\{e_1 + e_3, e_2\}$, $\{e_1 + e_3, e_4\}$, $\{e_1 + e_3, e_2, e_4\}$	$A_{3,3} \oplus A_1$
$\{e_1 - e_3\}$, $\{e_1 - e_3, e_2\}$, $\{e_1 - e_3, e_4\}$, $\{e_1 - e_3, e_2, e_4\}$	$A_{3,4} \oplus A_1$
$\{e_1 + xe_3\}$, $\{e_1 + xe_3, e_2\}$, $\{e_1 + xe_3, e_4\}$, $\{e_1 + xe_3, e_2, e_4\}$	$A_{3,5}^x \oplus A_1$, $0 < \lvert x \rvert < 1$ $A_{3,5}^{x^{-1}} \oplus A_1$, $\lvert x \rvert > 1$
$\{e_1 + \epsilon e_4\}$, $\{e_2 + \epsilon e_3\}$, $\{e_1 + \epsilon e_4, e_2\}$, $\{e_3 + \epsilon e_2, e_4\}$	$A_{4,3}$
$\{e_1 + e_3, e_2 + \epsilon e_4\}$	$A_{4,9}^0$

TABLE III. Contractions of $A_{3,1} \oplus A_1$. The range of parameters: $-\infty < x, y < \infty$, $0 \leqslant \phi < \pi$.

Nonzero Lie brackets $[e_2, e_3] = e_1$	
Subalgebra representative	Contracted algebra
$\{e_1 + xe_4\}$, $\{e_4\}$, $\{e_1, e_4 + x(e_2\cos\phi + e_3\sin\phi)\}$, $\{e_1, e_2\cos\phi + e_3\sin\phi\}$, $\{e_1, e_2\cos\phi + e_3\sin\phi, e_4\}$	$4A_1$
$\{e_2\cos\phi + e_3\sin\phi + xe_4\}$, $\{e_4, e_2\cos\phi + e_3\sin\phi\}$, $\{e_1 + xe_4, e_2\cos\phi + e_3\sin\phi\}$, $x \neq 0$, $\{e_2 + xe_4, e_3 + ye_4, e_1\}$	$A_{3,1} \oplus A_1$

TABLE IV. Contractions of $A_{3,2}\oplus A_1$. The range of parameters: $-\infty<x<\infty$, $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_1,e_3]=e_1$, $[e_2,e_3]=e_1+e_2$
Subalgebra representative	Contracted algebra
$\{e_1\}$, $\{e_4\}$, $\{e_1,e_2\}$, $\{e_1,e_4\}$, $\{e_1,e_2,e_4\}$	$4A_1$
$\{e_1+\epsilon e_4\}$, $\{e_2+xe_4\}$, $\{e_1+xe_4,e_2\}$, $x\neq 0$, $\{e_1,e_2+\epsilon e_4\}$, $\{e_2,e_4\}$	$A_{3,1}\oplus A_1$
$\{e_3+xe_4\}$, $\{e_3,e_4\}$, $\{e_3+xe_4,e_1,e_2\}$	$A_{3,2}\oplus A_1$
$\{e_3+xe_4,e_1\}$, $\{e_1,e_3,e_4\}$	$A_{3,3}\oplus A_1$

TABLE V. Contractions of $A_{3,3}\oplus A_1$. The range of parameters: $-\infty<x<\infty$, $0\leqslant\phi<\pi$, $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_1,e_3]=e_1$, $[e_2,e_3]=e_2$
Subalgebra representative	Contracted algebra
$\{e_1\cos\phi+e_2\sin\phi\}$, $\{e_4\}$, $\{e_1,e_2\}$, $\{e_1\cos\phi+e_2\sin\phi,e_4\}$, $\{e_1,e_2,e_4\}$	$4A_1$
$\{e_1\cos\phi+e_2\sin\phi+\epsilon e_4\}$, $\{e_1,e_2+\epsilon e_4\}$, $\{e_1+\epsilon e_4,e_2+xe_1\}$	$A_{3,1}\oplus A_1$
$\{e_3+xe_4\}$, $\{e_3,e_4\}$, $\{e_3+xe_4,e_1\cos\phi+e_2\sin\phi\}$, $\{e_3,e_4,e_1\cos\phi+e_2\sin\phi\}$, $\{e_3+xe_4,e_1,e_2\}$	$A_{3,3}\oplus A_1$

TABLE VI. Contractions of $A_{3,4}\oplus A_1$. The range of parameters: $-\infty<x<\infty$, $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_1,e_3]=e_1$, $[e_2,e_3]=-e_2$
Subalgebra representative	Contracted algebra
$\{e_1\}$, $\{e_2\}$, $\{e_4\}$, $\{e_1,e_2\}$, $\{e_1,e_4\}$, $\{e_2,e_4\}$, $\{e_1,e_2,e_4\}$	$4A_1$
$\{e_1+\epsilon e_2+xe_4\}$, $\{e_1+\epsilon e_4\}$, $\{e_2+\epsilon e_4\}$, $\{e_1+\epsilon e_2,e_4\}$, $\{e_1,e_2+\epsilon e_4\}$, $\{e_1+\epsilon e_4,e_2+xe_1\}$	$A_{3,1}\oplus A_1$
$\{e_3+xe_4\}$, $\{e_3,e_4\}$, $\{e_3+xe_4,e_1\}$, $\{e_3+xe_4,e_2\}$, $\{e_1,e_3,e_4\}$, $\{e_2,e_3,e_4\}$, $\{e_3+xe_4,e_1,e_2\}$	$A_{3,4}\oplus A_1$

TABLE VII. Contractions of $A_{3,5}^a\oplus A_1$, $0<|a|<1$. The range of parameters: $-\infty<x<\infty$, $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_1,e_3]=e_1$, $[e_2,e_3]=ae_2$
Subalgebra representative	Contracted algebra
$\{e_1\}$, $\{e_2\}$, $\{e_4\}$, $\{e_1,e_2\}$, $\{e_1,e_4\}$, $\{e_2,e_4\}$, $\{e_1,e_2,e_4\}$	$4A_1$
$\{e_1+\epsilon e_4\}$, $\{e_2+\epsilon e_4\}$, $\{e_1+\epsilon e_2+xe_4\}$, $\{e_1+\epsilon e_2,e_4\}$, $\{e_1,e_2+\epsilon e_4\}$, $\{e_1+\epsilon e_4,e_2+xe_4\}$	$A_{3,1}\oplus A_1$
$\{e_3+xe_4\}$, $\{e_3,e_4\}$, $\{e_3+xe_4,e_1\}$, $\{e_3+xe_4,e_2\}$, $\{e_1,e_3,e_4\}$, $\{e_2,e_3,e_4\}$, $\{e_3+xe_4,e_1,e_2\}$	$A_{3,5}^a\oplus A_1$

TABLE VIII. Contractions of $A_{3,6}\oplus A_1$. The range of parameters: $-\infty<x,y<\infty$.

Nonzero Lie brackets	$[e_1,e_3]=-e_2$, $[e_2,e_3]=e_1$
Subalgebra representative	Contracted algebra
$\{e_4\}$, $\{e_1,e_2\}$, $\{e_1,e_2,e_4\}$	$4A_1$
$\{e_1+xe_4\}$, $x\geqslant 0$, $\{e_1+xe_4,e_2\}$, $x>0$, $\{e_1,e_4\}$	$A_{3,1}\oplus A_1$
$\{e_3+ye_4\}$, $\{e_3,e_4\}$, $\{e_3+xe_4,e_1,e_2\}$	$A_{3,6}\oplus A_1$

TABLE IX. Contractions of $A_{3,7}^a\oplus A_1$, $a>0$. The range of parameters: $-\infty<x,y<\infty$.

Nonzero Lie brackets	$[e_1,e_3]=ae_1-e_2$, $[e_2,e_3]=e_1+ae_2$
Subalgebra representative	Contracted algebra
$\{e_4\}$, $\{e_1,e_2\}$, $\{e_1,e_2,e_4\}$	$4A_1$
$\{e_1+xe_4\}$, $x\geqslant 0$, $\{e_1,e_4\}$, $\{e_1+xe_4,e_2\}$, $x>0$	$A_{3,1}\oplus A_1$
$\{e_3+ye_4\}$, $\{e_3,e_4\}$, $\{e_3+xe_4,e_1,e_2\}$	$A_{3,7}^a\oplus A_1$

TABLE X. Contractions of $A_{3,8}\oplus A_1$. The range of parameters: $-\infty<x,y<\infty$, $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_3,e_1]=2e_2$, $[e_1,e_2]=e_1$, $[e_2,e_3]=e_3$
Subalgebra representative	Contracted algebra
$\{e_4\}$	$4A_1$
$\{e_1\}$, $\{e_1,e_4\}$	$A_{3,1}\oplus A_1$
$\{e_2+xe_4\}$, $x\geqslant 0$, $\{e_2,e_4\}$, $\{e_1,e_2\}$, $\{e_1,e_2,e_4\}$	$A_{3,4}\oplus A_1$
$\{e_1-e_3\}$, $\{e_1-e_3,e_4\}$	$A_{3,6}\oplus A_1$
$\{e_1,e_2,e_3\}$	$A_{3,8}\oplus A_1$
$\{e_1-e_3+ye_4\}$, $y\neq 0$, $\{e_1+\epsilon e_4\}$	$A_{4,1}$
$\{e_2+xe_4,e_1\}$, $x\neq 0$	$A_{4,8}$

TABLE XI. Contractions of $A_{3,9}\oplus A_1$. The range of parameter: $0\leqslant x<\infty$.

Nonzero Lie brackets	$[e_1,e_2]=e_3$, $[e_2,e_3]=e_1$, $[e_3,e_1]=e_2$
Subalgebra representative	Contracted algebra
$\{e_4\}$	$4A_1$
$\{e_1+xe_4\}$, $\{e_1,e_4\}$	$A_{3,6}\oplus A_1$
$\{e_1,e_2,e_3\}$	$A_{3,9}\oplus A_1$

TABLE XII. Contractions of $A_{4,1}$. The range of parameter: $-\infty<x<\infty$.

Nonzero Lie brackets	$[e_2,e_4]=e_1$, $[e_3,e_4]=e_2$
Subalgebra representative	Contracted algebra
$\{e_1\}$, $\{e_1,e_2\}$, $\{e_1,e_2,e_3\}$	$4A_1$
$\{e_2\}$, $\{e_3+xe_1\}$, $\{e_1,e_3\}$, $\{e_2,e_3+xe_1\}$, $\{e_1,e_4+xe_3\}$, $\{e_4+xe_3,e_1,e_2\}$	$A_{3,1}\oplus A_1$
$\{e_4+xe_3\}$	$A_{4,1}$

TABLE XIII. Contractions of $A_{4,2}^a$, $a\neq 0,1$. The range of parameters: $-\infty<x<\infty$, $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_1,e_4]=ae_1$, $[e_2,e_4]=e_2$, $[e_3,e_4]=e_2+e_3$
Subalgebra representative	Contracted algebra
$\{e_1\}$, $\{e_2\}$, $\{e_1,e_2\}$, $\{e_2,e_3\}$, $\{e_1,e_2,e_3\}$	$4A_1$
$\{e_1+\epsilon e_2\}$, $\{e_3+xe_1\}$, $\{e_1+xe_2,e_3\}$, $\{e_1+\epsilon e_3,e_2\}$	$A_{3,1}\oplus A_1$
$\{e_4\}$, $\{e_1,e_4\}$, $\{e_2,e_3,e_4\}$	$A_{4,2}^a$
$\{e_2,e_4\}$, $\{e_1,e_2,e_4\}$	$A_{4,5}^{a^{-1},a^{-1}}$, $\lvert a\rvert>1$
$\{e_2,e_4\}$, $\{e_1,e_2,e_4\}$	$A_{4,5}^{a,1}$, $\lvert a\rvert<1$
$\{e_2,e_4\}$, $\{e_1,e_2,e_4\}$	$A_{4,5}^{-1,1}$, $a=-1$

TABLE XIV. Contractions of $A^1_{4,2}$. The range of parameters: $-\infty < x < \infty$, $0 \le \phi < \pi$.

Nonzero Lie brackets	$[e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$
Subalgebra representative	Contracted algebra
$\{e_1 \cos\phi + e_2 \sin\phi\}, \{e_1, e_2\}, \{e_2, e_3 + xe_1\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_3 + xe_1\}, \{e_1 + xe_2, e_3\}$	$A_{3,1} \oplus A_1$
$\{e_4\}, \{e_4, e_1 \cos\phi + e_2 \sin\phi\}, \phi \neq \frac{1}{2}\pi, \{e_2, e_4, e_3 + xe_1\}$	$A^1_{4,2}$
$\{e_2, e_4\}, \{e_1, e_2, e_4\}$	$A^{1,1}_{4,5}$

TABLE XV. Contractions of $A_{4,3}$. The range of the parameters: $-\infty < x < \infty$, $\epsilon = \pm 1$.

Nonzero Lie brackets	$[e_1, e_4] = e_1, [e_3, e_4] = e_2$
Subalgebra representative	Contracted algebra
$\{e_1\}, \{e_2\}, \{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_2, e_4 + xe_3\}, \{e_1, e_2, e_4 + xe_3\}$	$A_2 \oplus 2A_1$
$\{e_1 + \epsilon e_2\}, \{e_3 + xe_1\}, \{e_1 + xe_2, e_3\}, \{e_2, e_3 + \epsilon e_1\}$	$A_{3,1} \oplus A_1$
$\{e_4 + xe_3\}, \{e_1, e_4 + xe_3\}, \{e_2, e_3, e_4\}$	$A_{4,3}$

TABLE XVI. Contractions of $A_{4,4}$. The range of parameter: $-\infty < x < \infty$.

Nonzero Lie brackets	$[e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3$
Subalgebra representative	Contracted algebra
$\{e_1\}, \{e_1, e_2\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_1 + xe_3\}, x \neq 0, \{e_2\}, \{e_3\}, \{e_1, e_3\}, \{e_1 + xe_3, e_2\}, x \neq 0, \{e_2, e_3\}$	$A_{3,1} \oplus A_1$
$\{e_1, e_4\}, \{e_1, e_2, e_4\}$	$A^1_{4,2}$
$\{e_4\}$	$A_{4,4}$

TABLE XVII. Contractions of $A^{a,b}_{4,5}$, $-1 \le a < b < 1$, $ab \neq 0$. The range of parameters: $-\infty < x < \infty$, $\epsilon = \pm 1$.

Nonzero Lie brackets	$[e_1, e_4] = e_1, [e_2, e_4] = ae_2, [e_3, e_4] = be_3$
Subalgebra representative	Contracted algebra
$\{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_1 + \epsilon e_3\}, \{e_2 + \epsilon e_3\}, \{e_1 + \epsilon e_2 + xe_3\}, x \neq 0, \{e_1, e_2 + \epsilon e_3\}, \{e_2, e_1 + \epsilon e_3\}, \{e_3, e_1 + \epsilon e_2\}, \{e_1 + \epsilon e_3, e_2 + xe_3\}, x \neq 0$	$A_{3,1} \oplus A_1$
$\{e_4\}, \{e_1, e_4\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}$	$A^{a,b}_{4,5}$

TABLE XVIII. Contractions of $A^{a,a}_{4,5}$, $-1 \le a < 1$, $a \neq 0$. The range of parameters: $-\infty < x < \infty$, $0 \le \phi < \pi$, $\epsilon = \pm 1$.

Nonzero Lie brackets	$[e_1, e_4] = e_1, [e_2, e_4] = ae_2, [e_3, e_4] = ae_3$
Subalgebra representative	Contracted algebra
$\{e_1\}, \{e_2 \cos\phi + e_3 \sin\phi\}, \{e_1, e_2 \cos\phi + e_3 \sin\phi\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_1 + \epsilon e_3\}, \{e_1 + \epsilon e_2 + xe_3\}, \{e_3, e_1 + \epsilon e_2\}, \{e_1 + \epsilon e_3, e_2 + xe_3\}$	$A_{3,1} \oplus A_1$
$\{e_4\}, \{e_1, e_4\}, \{e_4, e_2 \cos\phi + e_3 \sin\phi\}, \{e_2, e_3, e_4\}, \{e_1, e_4, e_2 \cos\phi + e_3 \sin\phi\}$	$A^{a,a}_{4,5}$

TABLE XIX. Contractions of $A^{a,1}_{4,5}$, $-1 \le a < 1$, $a \neq 0$. The range of parameters: $-\infty < x < \infty$, $0 \le \phi < \pi$, $\epsilon = \pm 1$.

Nonzero Lie brackets	$[e_1, e_4] = e_1, [e_2, e_4] = ae_2, [e_3, e_4] = e_3$
Subalgebra representative	Contracted algebra
$\{e_2\}, \{e_1 \cos\phi + e_3 \sin\phi\}, \{e_1 \cos\phi + e_3 \sin\phi, e_2\}, \{e_1, e_3\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_2 + \epsilon e_3\}, \{e_1 + \epsilon e_2 + xe_3\}, \{e_3, e_1 + \epsilon e_2\}, \{e_1 + xe_3, e_2 + \epsilon e_3\}$	$A_{3,1} \oplus A_1$
$\{e_4\}, \{e_2, e_4\}, \{e_4, e_1 \cos\phi + e_3 \sin\phi\}, \{e_1, e_3, e_4\}, \{e_2, e_1 \cos\phi + e_3 \sin\phi, e_4\}$	$A^{a,1}_{4,5}$

TABLE XX. Contractions of $A^{1,1}_{4,5}$. The range of parameters: $-\infty < x, y < \infty$.

Nonzero Lie brackets	$[e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3$
Subalgebra representative	Contracted algebra
$\{e_1 + xe_2 + ye_3\}, \{e_2 + xe_3\}, \{e_3\}, \{e_1 + xe_3, e_2 + ye_3\}, \{e_1 + xe_2, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_4\}, \{e_4, e_1 + xe_2 + ye_3\}, \{e_4, e_2 + xe_3\}, \{e_3, e_4\}, \{e_4, e_1 + xe_3, e_2 + ye_3\}, \{e_4, e_1 + xe_2, e_3\}, \{e_2, e_3, e_4\}$	$A^{1,1}_{4,5}$

TABLE XXI. Contractions of $A^{a,b}_{4,6}$, $a \neq 0$, $b \ge 0$. The range of parameter: $0 \le x < \infty$.

Nonzero Lie brackets	$[e_1, e_4] = ae_1, [e_2, e_4] = be_2 - e_3, [e_3, e_4] = e_2 + be_3$
Subalgebra representative	Contracted algebra
$\{e_1\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}$	$4A_1$
$\{e_1 + xe_3\}, x > 0, \{e_3\}, \{e_1 + xe_3, e_2\}$	$A_{3,1} \oplus A_1$
$\{e_4\}, \{e_1, e_4\}, \{e_2, e_3, e_4\}$	$A^{a,b}_{4,6}$

TABLE XXII. Contractions of $A_{4,7}$.

Nonzero Lie brackets	$[e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_2, e_3] = e_1, [e_3, e_4] = e_2 + e_3$
Subalgebra representative	Contracted algebra
$\{e_1\}, \{e_1, e_2\}$	$4A_1$
$\{e_2\}, \{e_1, e_3\}, \{e_1, e_2, e_3\}$	$A_{3,1} \oplus A_1$
$\{e_3\}$	$A_{4,1}$
$\{e_4\}, \{e_1, e_4\}$	$A^2_{4,2}$
$\{e_1, e_2, e_4\}$	$A^{1/2,1/2}_{4,5}$
$\{e_2, e_4\}$	$A^1_{4,9}$

TABLE XXIII. Contractions of $A_{4,8}$. The range of parameters: $-\infty < x < \infty$, $\epsilon = \pm 1$.

Nonzero Lie brackets	$[e_2, e_3] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = -e_3$
Subalgebra representative	Contracted algebra
$\{e_1\}, \{e_1, e_2\}, \{e_1, e_3\}$	$4A_1$
$\{e_2\}, \{e_3\}, \{e_1, e_2 + \epsilon e_3\}, \{e_1, e_2, e_3\}$	$A_{3,1} \oplus A_1$
$\{e_4 + xe_1\}, \{e_1, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}$	$A_{3,4} \oplus A_1$
$\{e_2 + \epsilon e_3\}$	$A_{4,1}$
$\{e_4 + xe_1, e_2\}, \{e_4 + xe_1, e_3\}$	$A_{4,8}$

TABLE XXIV. Contractions of $A_{4,9}^{b}$, $0<|b|<1$. The range of parameter: $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_2,e_3]=e_1,[e_1,e_4]=(1+b)e_1$ $[e_2,e_4]=e_2,[e_3,e_4]=be_3$
Subalgebra representative	Contracted algebra
$\{e_1\},\{e_1,e_2\},\{e_1,e_3\}$	$4A_1$
$\{e_2\},\{e_3\},\{e_1,e_2+\epsilon e_3\},\{e_1,e_2,e_3\}$	$A_{3,1}\oplus A_1$
$\{e_2+\epsilon e_3\}$	$A_{4,1}$
$\{e_1,e_3,e_4\}, b=-\frac{1}{2}$	$A_{4,5}^{-1/2,1/2}$
$\{e_4\},\{e_1,e_4\},\{e_1,e_2,e_4\},\{e_1,e_3,e_4\}, b\neq-\frac{1}{2}$	$A_{4,5}^{(1+b)^{-1},b(1+b)^{-1}}, b>0$ $A_{4,5}^{b,1+b}, b<0$
$\{e_2,e_4\},\{e_3,e_4\}$	$A_{4,9}^{b}$

TABLE XXV. Contractions of $A_{4,9}^{1}$. The range of parameters: $0\leq\phi<\pi$.

Nonzero Lie brackets	$[e_2,e_3]=e_1,[e_1,e_4]=2e_1$ $[e_2,e_4]=e_2,[e_3,e_4]=e_3$
Subalgebra representative	Contracted algebra
$\{e_1\},\{e_1,e_2\cos\phi+e_3\sin\phi\}$	$4A_1$
$\{e_2\cos\phi+e_3\sin\phi\},\{e_1,e_2,e_3\}$	$A_{3,1}\oplus A_1$
$\{e_4\},\{e_1,e_4\},\{e_1,e_4,e_2\cos\phi+e_3\sin\phi\}$	$A_{4,5}^{1/2,1/2}$
$\{e_4,e_2\cos\phi+e_3\sin\phi\}$	$A_{4,9}^{1}$

TABLE XXVI. Contractions of $A_{4,9}^{0}$. The range of parameters: $-\infty<x<\infty$, $\epsilon=\pm 1$.

Nonzero Lie brackets	$[e_2,e_3]=e_1,[e_1,e_4]=e_1,[e_2,e_4]=e_2$
Subalgebra representative	Contracted algebra
$\{e_1\},\{e_1,e_2\},\{e_1,e_3\}$	$4A_1$
$\{e_2\},\{e_3\},\{e_1,e_2+\epsilon e_3\},\{e_1,e_2,e_3\}$	$A_{3,1}\oplus A_1$
$\{e_4+xe_3\}, x\neq 0$, $\{e_1,e_2,e_4+xe_3\}, x\neq 0$	$A_{3,2}\oplus A_1$
$\{e_4\},\{e_1,e_4+xe_3\},\{e_1,e_3,e_4\},\{e_1,e_2,e_4\}$	$A_{3,3}\oplus A_1$
$\{e_2+\epsilon e_3\}$	$A_{4,1}$
$\{e_3,e_4\},\{e_2,e_4\}$	$A_{4,9}^{0}$

TABLE XXVII. Contractions of $A_{4,10}$. The range of parameter: $-\infty<x<\infty$.

Nonzero Lie brackets	$[e_2,e_3]=e_1,[e_2,e_4]=-e_3,[e_3,e_4]=e_2$
Subalgebra representative	Contracted algebra
$\{e_1\}$	$4A_1$
$\{e_1,e_2\},\{e_1,e_2,e_3\}$	$A_{3,1}\oplus A_1$
$\{e_4+xe_1\},\{e_1,e_4\}$	$A_{3,6}\oplus A_1$
$\{e_2\}$	$A_{4,1}$

TABLE XXVIII. Contractions of $A_{4,11}^{a}$, $a>0$.

Nonzero Lie brackets	$[e_2,e_3]=e_1$, $[e_1,e_4]=2ae_1$ $[e_2,e_4]=ae_2-e_3,[e_3,e_4]=e_2+ae_3$
Subalgebra representative	Contracted algebra
$\{e_1\}$	$4A_1$
$\{e_1,e_2\},\{e_1,e_2,e_3\}$	$A_{3,1}\oplus A_1$
$\{e_2\}$	$A_{4,1}$
$\{e_4\},\{e_1,e_4\}$	$A_{4,6}^{2a,a}$

TABLE XXIX. Contractions of $A_{4,12}$. The range of parameter: $-\infty<x<\infty$.

Nonzero Lie brackets	$[e_1,e_3]=e_1$, $[e_2,e_3]=e_2$ $[e_1,e_4]=-e_2,[e_2,e_4]=e_1$
Subalgebra representative	Contracted algebra
$\{e_1,e_2\}$	$4A_1$
$\{e_1\}$	$A_{3,1}\oplus A_1$
$\{e_3\},\{e_1,e_2,e_3\}$	$A_{3,3}\oplus A_1$
$\{e_4\},\{e_1,e_2,e_4\}$	$A_{3,6}\oplus A_1$
$\{e_4+xe_3\}, x\neq 0,\{e_1,e_2,e_4+xe_3\}, x\neq 0$	$A_{3,7}^{\vert x\vert}\oplus A_1$
$\{e_1,e_3\}$	$A_{4,9}^{0}$
$\{e_3,e_4\}$	$A_{4,12}$

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Mesonic test fields and spacetime cohomology[a)]

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We prove the following theorem: Classical spin-0 and -1 mesonic test fields that can be constructed on a given spacetime manifold determine at least some of its de Rham cohomological structure. We explore this result and give some examples. The extension of the present technique to higher spin is also discussed.

1. INTRODUCTION

Teitler[1] and Hestenes[2] have consistently used and developed the Clifford–Dirac algebra along Riesz's[3] geometrical interpretation in order to unify geometrical and physical concepts in classical field theory. Hestenes[4] has specifically suggested the search for a topologico–differential connection between fields and the underlying spacetime structure. In the present note we introduce one such bridge between physics and geometry with the help of Teitler's equation for spin-0 and -1 mesonic fields.[5] It has the form of some theorems that relate mesonic test fields defined on a spacetime manifold to the underlying space's de Rham cohomology. We also show that despite the fact that one has a Teitler-like equations for the gravitational field itself,[6] the present technique cannot be generalized to that equation. Actually it cannot be applied to spins higher than 1.

2. THE MAIN RESULTS

We start from Minkowski spacetime V, with a $+2$ metric $g_{\mu\nu}$ and an associated Clifford–Dirac algebra C_4 generated by the four anticommuting γ's subject to $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$. Teitler's equations are summarized in the Dirac-like equation

$$(\nabla + m)\Psi_i = 0, \quad i = 0, 1, \tag{2.1}$$

with $\nabla = \gamma^\mu \partial_\mu$, $\Psi_0 = -m\psi + \psi_\mu \gamma^\mu$, $\Psi_1 = -m\phi_\mu\gamma^\mu + \frac{1}{2}\phi_{\mu\nu}\gamma^{\mu\nu}$ (Ψ_0 describes a massive spin-0 field, Ψ_1 a massive spin-1 field, and $\Psi_0\gamma^5$ is Teitler's wavefunction for the pseudoscalar meson).

We can identify C_4 with its associated exterior algebra $\Lambda = \oplus\sum_{k=0}^4 \Lambda^k$.[7] This association induces an exterior-algebra representation of Dirac's operator $\nabla = d - \delta$, where d is the exterior derivative acting on Λ and δ the associated divergence. With the help of this notation and of $\Psi_{(0)0} = \psi$, $\Psi_{(0)1} = \psi_\mu\gamma^\mu$, $\Psi_{(1)1} = \phi_\mu\gamma^\mu$, $\Psi_{(1)2} = \phi_{\mu\nu}\gamma^{\mu\nu}$, the spin-0 equations implied by (2.1) are

$$\delta\Psi_{(0)1} = -m^2\Psi_{(0)0}, \tag{2.2}$$

$$d\Psi_{(0)1} = 0, \tag{2.3}$$

$$\Psi_{(0)1} = d\Psi_{(0)0}, \tag{2.4}$$

and the spin-1 are

$$\delta\Psi_{(1)2} = -m^2\Psi_{(1)1}, \tag{2.5}$$

$$d\Psi_{(1)2} = 0, \tag{2.6}$$

$$\Psi_{(1)2} = d\Psi_{(1)1}, \tag{2.7}$$

$$\delta\Psi_{(1)1} = 0. \tag{2.8}$$

When going from Minkowski spacetime V to a pseudo-Riemannian one, M also endowed with a $+2$ metric $g_{\mu\nu}$, $\nabla \mapsto \hat{\nabla} = \gamma^\mu\nabla_\mu$, ∇_μ being the covariant derivative with respect to $g_{\mu\nu}$ and $\hat{\nabla} = \hat{d} - \hat{\delta}$, with $\hat{d}^2 = \hat{\delta}^2 = 0$. Equations (2.1)–(2.8) keep their form but for the substitutions $\nabla \mapsto \hat{\nabla}$, $d \mapsto \hat{d}$, and $\delta \mapsto \hat{\delta}$. We have, however, no reason to assert now—on physical and on mathematical grounds—that the relationship between fields and potentials [Eqs. (2.4) and (2.7)] may be smoothly extended over the whole manifold: Being differential relations, the field equations are a purely local affair, and we can always have nonsmooth and sectionally continuous potentials (see Sec. 3 for an example; the same remark has been made by Goldberg in a similar setting[8]).

Now Eqs. (2.3) and (2.6) say that the fields $\Psi_{(0)1}$ and $\Psi_{(1)2}$ are 1- and 2-cocycles; we note their set, $\mathcal{Z}^1$ and $\mathcal{Z}^2$, respectively. Some of these cocycles will be coboundaries, that is, will have potentials smoothly defined all over the manifold; denote their collections $\mathcal{B}^1 \subset \mathcal{Z}^1$ and $\mathcal{B}^2 \subset \mathcal{Z}^2$. The kth order de Rham cohomology groups for spacetime, $k = 1, 2$, are then $\mathcal{D}^k = \mathcal{Z}^k/\mathcal{B}^k$.[9] To get $\mathcal{D}^3$ see that Eq. (2.1) applied to the wavefunction $\Psi_0\gamma^5$ for a pseudoscalar massless meson leads to

$$\hat{\delta}\Psi_{(0)3} = 0, \tag{2.9}$$

$$\hat{d}\Psi_{(0)3} = 0, \tag{2.10}$$

where $\Psi_{(0)3} = (-g)^{-1/2} * \Psi_{(0)1}$, $*$ being Hodge's star operator.[8] Clearly the set of all such cocycles is $\mathcal{Z}^3$, and from the obvious definition for $\mathcal{B}^3 \subset \mathcal{Z}^3$ one gets $\mathcal{D}^3 = \mathcal{Z}^3/\mathcal{B}^3$. The de Rham cohomology groups for M, $\mathcal{D}^1$, $\mathcal{D}^2$, and $\mathcal{D}^3$ have thus been obtained with the help of mesonic test fields defined on that spacetime manifold. $\mathcal{D}^0 = \mathbb{R}$[10] since we suppose M to be connected. $\mathcal{D}^4$ is left undetermined if M is noncompact; for compact spacetimes (which are usually ruled out on the ground of causality considerations[11]) $\mathcal{D}^4 = \mathbb{R}$.

We may summarize as follows the above reasoning:

Proposition 1: Let M be a four-dimensional Hausdorff manifold endowed with a $+2$ metric. Let $\mathcal{Z}^1$ be the space of all spin-0 mesonic test fields defined over M, $\mathcal{Z}^2$ the space of all spin-1 fields similarly defined, and $\mathcal{Z}^3$ the space of all zero-mass noninteracting pseudoscalar fields on M. Then:

(i) If M is noncompact, $\mathcal{Z}^1$, $\mathcal{Z}^2$, and $\mathcal{Z}^3$ determine the de Rham cohomology groups $\mathcal{D}^1$, $\mathcal{D}^2$, and $\mathcal{D}^3$ for M.

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(ii) if M is compact, $\mathcal{Z}^1$, $\mathcal{Z}^2$, and $\mathcal{Z}^3$ completely determine the de Rham cohomological structure $\mathcal{D}=\oplus\sum_{k=0}^{4}\mathcal{D}^k$ for M.

Suppose now that one compactifies M with the help of some standard technique—for instance, Alexandrov's,[12] which is equivalent to Penrose's[13] but for the identification of all infinite points. This can be conformally achieved as follows: Let $x_0\in M$ be a suitable reference-point; for any other point $x\in M$, let $d(x, x_0)$ be the extremal path length (with respect to M's Lorentzian metric) between x and x_0 (since M is path-connected, one has always such a value). For a fixed x_0, put $[d(x,x_0)]^{-2}=\Omega_0(x)$; Ω_0 is a mapping $M\to\mathbb{R}$, and $\Omega_0(x)=0$ over the pencil of lightlike geodesics with x_0 as its vertex. If we identify the set $\{x\in M \mid \Omega_0(x)=0\}$ with the infinite point x_∞, we get a compactification for M with Ω_0 as the conformal factor.

Let $\widetilde{M}$ be such a compactification, the tilde denoting here Ω_0-conformally changed objects on $\widetilde{M}$. As it is well-known, for noninteracting zero-mass mesonic test fields, Eqs. (2.2)—(2.10) become:

$$\tilde{\delta}\tilde{\Psi}_{(0)1}=0, \tag{2.11}$$

$$\tilde{d}\tilde{\Psi}_{(0)1}=0, \tag{2.12}$$

for spin-0 scalar fields;

$$\tilde{\delta}\tilde{\Psi}_{(1)2}=0, \tag{2.13}$$

$$\tilde{d}\tilde{\Psi}_{(1)2}=0, \tag{2.14}$$

for spin-1 fields;

$$\tilde{\delta}\tilde{\Psi}_{(0)3}=0, \tag{2.15}$$

$$\tilde{d}\tilde{\Psi}_{(0)3}=0, \tag{2.16}$$

for spin-0 pseudoscalar fields. The space $\widetilde{\mathcal{N}}^1$ of all conformally changed massless noninteracting spin-0 fields generates $\widetilde{\mathcal{D}}^1$; a similarly defined $\widetilde{\mathcal{N}}^2$ for spin-1 fields generates $\widetilde{\mathcal{D}}^2$, and $\widetilde{\mathcal{N}}^3$ generates $\widetilde{\mathcal{D}}^3$. Due to $\widetilde{M}$'s compactness and connectedness, $\widetilde{\mathcal{D}}^4=\widetilde{\mathcal{D}}^0=\mathbb{R}$. We have thus

Proposition 2: Let $\widetilde{M}$ be a conformal compactification for M. Then $\widetilde{\mathcal{N}}^1$, $\widetilde{\mathcal{N}}^2$, and $\widetilde{\mathcal{N}}^3$ completely determine the de Rham cohomology for M.

Observe that this compactification obviously and drastically changes the manifold's topological properties. The compactification induced by the inverse Riemann mapping $\rho: S^2\to\mathbb{R}^2$ changes a manifold that admits a Lorentz metric ($\mathbb{R}^2$) into one that does not (S^2, see Ref. 14), since the Euler characteristic $\chi(S^2)=2\neq 0$. On the contrary the compactification of a cylindrical spacetime $S^3\times\mathbb{R}$ induced by the inverse map $\rho: S^3\times S^1\to S^3\times\mathbb{R}$ leads to one that admits a Lorentz metric, for $\chi(S^3\times S^1)=0$.

If we now write b_k for the dimension of $\widetilde{\mathcal{D}}^k$, we get the Euler—Poincaré formula[15]:

$$\chi(\widetilde{M})=\sum_{k=0}^{4}(-1)^k b_k, \tag{2.17}$$

b_k, the kth Betti number of $\widetilde{M}$, is the number of linearly independent classes of cocycles on $\widetilde{M}$. It is thus an integer. Then,

Proposition 3: If $\widetilde{M}$ is orientable and admits a Lorentz metric, then it has at least one class of nonbounding cocycles that represent spin-0 massless fields. If there are nonbounding zero-mass spin-1 fields, their Betti number $b_2\geq 2$.

Proof: Since $\widetilde{M}$ admits a Lorentz metric, $\chi(\widetilde{M})=0 =b_0-b_1+b_2-b_3+b_4$. But $b_0=b_4=1$. As $b_1=b_3$ as a consequence of the Poincaré duality,[10] we have $2b_1=2+b_2$. For $b_2=0$, $b_1=1$. Concerning the second statement, see that $b_2=2$ is the smallest nontrivial solution for the preceding equations.

This result leads to an immediate corollary:

Corollary: Propostion 3 is valid for a compact space spacetime M.

And also,

Proposition 4: Let M (resp. $\widetilde{M}$) be a spacetime (resp. compactified spacetime) whose de Rham cohomological structure $\mathcal{D}$ ($\widetilde{\mathcal{D}}$) is given. Then every class of scalar, pseudoscalar and vector mesonic test fields defined all over M (resp. $\widetilde{M}$) must be compatible with $\mathcal{D}$ (resp. $\widetilde{\mathcal{D}}$).

That is, Propositions 1—3 are "if and only if" conditions. Many such results on test mesonic fields can be proved from standard theorems in de Rham cohomology; we especially note that the fields given by Eqs. (2.11)—(2.16) are harmonic with respect to the pseudo-Riemannian metric $g_{\mu\nu}$; there are thus further general generalizations of the above results along the theory of harmonic forms.

3. EXAMPLES AND APPLICATIONS

In the present section we give some examples in order to show clearly the difference between cocycles and coboundaries. We also discuss the cohomological properties of some simple spacetimes.

Let us consider the cylinder C embedded in $\mathbb{R}^3$ with rectangular coordinates and given by the equation $x^2+y^2=1$. Define the two 1-forms,

$$\alpha=(x\,dx+y\,dy)\,/(x^2+y^2), \tag{3.1}$$

$$\beta=(x\,dy-y\,dx)/(x^2+y^2). \tag{3.2}$$

We will only consider the restrictions $\alpha_C=\alpha\,|\,C$ and $\beta_C=\beta\,|\,C$. Trivially $d\alpha_C=d\beta_C=0$, that is, both are cocycles. We now show that while α_C is a coboundary, that is, admits a continuous and adequately differentiable potential ϕ_C over C such that $\alpha_C=d\phi_C$, there is no such construction for β_C. If we put $z=x+iy$, $\alpha_C=d(\mathrm{Re}\log z)\,|\,C$ and $\beta_C=d(\mathrm{Im}\log z)\,|\,C$, Re and Im denoting real and imaginary parts of the function. Re $\log z$ C is a 0-form over C, as a single-valued function of z; that, however, is not the case with Im $\log z\,|\,C$, which is not single-valued. The potential ϕ_C for α_C is then $\phi_C=\mathrm{Re}\log z$ all over C (actually over $\mathbb{R}^3-\zeta$, where ζ is the z axis).

Let now Γ be the closed path on C given by the equations $x^2+y^2=1$, $z=0$. It is easily verified that

$$\int_\Gamma \alpha_C=0$$

while

$$\int_\Gamma \beta_C = 2\pi.$$

In an elementary language we say that α_C is a gradient; its circulation over a closed path must be zero. As β_C is no gradient its circulation has a nonzero value for a conveniently chosen path.

The cylinder C is topologically $S^1\times\mathbb{R}$, where S^1 is the 1-sphere. From the Künneth isomorphism,[10]

$$\mathcal{D}(M\times N)\cong\mathcal{D}(M)\hat{\otimes}\mathcal{D}(N), \tag{3.3}$$

where the dimension of at least one of the cohomology algebras $\mathcal{D}(M)$, $\mathcal{D}(N)$, is finite, M and N being differentiable manifolds and $\hat{\otimes}$ being the skew-symmetric tensor product, and with the help of the relations[10]

$$\mathcal{D}^0(\mathbb{R}^n)=\mathbb{R}, \tag{3.4}$$

$$\mathcal{D}^p(\mathbb{R}^n)=0,\quad p\neq 0, \tag{3.5}$$

$$\mathcal{D}^0(S^n)=\mathcal{D}^n(S^n)=\mathbb{R}, \tag{3.6}$$

$$\mathcal{D}^p(S^n)=0,\quad 1\leq p\leq n-1, \tag{3.7}$$

one can establish that

$$\mathcal{D}^0(S^1\times\mathbb{R})=\mathbb{R}, \tag{3.8}$$

$$\mathcal{D}^1(S^1\times\mathbb{R})=\mathbb{R}, \tag{3.9}$$

$$\mathcal{D}^2(S^1\times\mathbb{R})=0, \tag{3.10}$$

Being one-dimensional, the cohomology group $\mathcal{D}^1(S^1\times\mathbb{R})$ admits just one cohomology class, which is represented by the cocycle β_C.

For a more realistic example let us consider a noncompact manifold C' embedded in $\mathbb{R}^5$ with topology $S^2\times\mathbb{R}^2$ described by the polar equation $\rho=1$ (in $\mathbb{R}^3$ with spherical coordinates ρ, θ, φ), and the remaining rectangular coordinates u, $v\in\mathbb{R}$. Consider now the 2-form given by

$$\omega_{C'}=\rho^{-2}\sin\theta\, d\theta\wedge d\varphi \tag{3.11}$$

and restricted to C'. Over the close surface Σ defined by $\rho=1$, $u=v=0$ (which is a 2-cycle) one has

$$\int_\Sigma \omega_{C'}=4\pi. \tag{3.12}$$

This integral must be zero when taken over a close 2-surface if its argument is to be a coboundary.[16] But as $\omega_{C'}$ represents the only cohomology class in the group $\mathcal{D}^2(S^2\times\mathbb{R}^2)=\mathbb{R}$, it is nonzero. The manifold's cohmological structure is given by

$$\mathcal{D}^0(S^2\times\mathbb{R}^2)=\mathcal{D}^2(S^2\times\mathbb{R}^2)=\mathbb{R}, \tag{3.13}$$

$$\mathcal{D}^1(S^2\times\mathbb{R}^2)=\mathcal{D}^3(S^2\times\mathbb{R}^2)=\mathcal{D}^4(S^2\times\mathbb{R}^2)=0. \tag{3.14}$$

Another interesting example is given by a cylindrical spacetime M with topology $S^3\times\mathbb{R}$ (the Einstein static universe and de Sitter spacetime have this structure). Its de Rham cohomology is given by

$$\mathcal{D}^0(S^3\times\mathbb{R})=\mathcal{D}^3(S^3\times\mathbb{R})=\mathbb{R}, \tag{3.15}$$

$$\mathcal{D}^i(S^3\times\mathbb{R})=0,\quad i\neq 0,\ 3. \tag{3.16}$$

Now let us get a conformal compactification for M. Given the projection $\pi: S^3\times\mathbb{R}\to S^3$, one can suppose the fiber over any $x\subset S^3$, $\pi^{-1}(x)$, to be a timelike path belonging to the manifold. For a fixed cross section $f: S^3\to S^3\times\mathbb{R}$, one can define the conformal factor $\Omega_0(x)$ as a function of $d(x,x_0)$, length of the fiber segment between $x_0\in f(S^3)$ and $x\in\pi^{-1}(x_0)$, namely $\Omega_0(x)=[d(x,x_0)]^{-2}$. such a length being constructed with the help of M's Lorentzian metric. The compactified spacetime $\tilde{M}\cong S^3\times S^1$, and has the de Rham cohomology

$$\mathcal{D}^i(S^3\times S^1)=\mathbb{R}\quad i=0,\ 1,\ 3,\ 4, \tag{3.17}$$

$$\mathcal{D}^2(S^3\times S^1)=0. \tag{3.18}$$

As we know from Proposition 3, Sec. 2, $\tilde{M}$ admits a nontrivial cocycle class represented by massless spin-0 test fields, and no spin-1 nonbounding cocycles.

4. DIFFICULTIES CONCERNING EXTENSIONS OF THE PRESENT TECHNIQUE

Can we get a similar connection between higher spin and the spacetime topology? We have shown elsewhere[17] that higher-spin fields can be also derived from a Dirac-like equation defined over a Cartesian product $\times V$ of copies of Minkowski space. When passing to the pseudo-Riemannian structure of a spacetime M and its Cartesian products $\times M$ with itself, the interlocking of fields and potentials through an exterior derivative breaks down due to the noncommutativity of the covariant derivative operator. But one still has a Dirac-like equation for massless higher-spin fields interacting with the gravitational field, and for such an equation there is no fixed set of subsidiary conditions due to the gauge freedom associated with zero-mass fields; the spin-2 case has been described by one of us[6] and will be considered here. With the help of the notation of[6] one may see that the field variable $\mathcal{R}$ (the Riemann–Christoffel tensor defined on the algebra $C_4\otimes C_4$) is a 4-form over the product $M\times M$; more precisely $\mathcal{R}\in\wedge^2\odot\wedge^2$, where $\wedge^2$ belongs to M's exterior bundle and $\odot$ is a symmetrized tensor product. But the differential operators $e_1^\mu\nabla_\mu=\tilde{\nabla}_1=\tilde{d}_1-\tilde{\delta}_1$ are no cohomology operators, that is, $\tilde{d}_1^2\neq 0$ and $\tilde{\delta}_1^2\neq 0$, and the field equations

$$\tilde{\delta}_1\mathcal{R}=0, \tag{4.1}$$

$$\tilde{d}_1\mathcal{R}=0, \tag{4.2}$$

cannot be interpreted as we did in the spin-0 and -1 cases. Actually $\mathcal{R}$ will *never* be a cocycle, for the Bianchi identities (4.2) may be written $\hat{d}_1\mathcal{R}+\Gamma=0$, where $\hat{d}_1$ is the partial exterior derivative acting on first factors in a tensor product[9] and Γ is a linear combination of Christoffel affinities. $\mathcal{R}$ is also a very particular object a symmetrized element of one of the various tensor products that add up to the module of 4-forms over $M\times M$, viz.,

$$\wedge^4(M\times M)\cong\wedge^0(M)\otimes\wedge^4(M)\oplus\wedge^1(M)\otimes\wedge^3(M)$$
$$\oplus\wedge^2(M)\otimes\wedge^2(M)\oplus\wedge^3(M)\otimes\wedge^1(M).$$

Being of a too restricted nature, it will not be useful as a source of information about the manifold's topological structure associated with its exterior bundle. This reasoning easily extends to any zero-mass integral spin minimally coupled to the gravitational field with spin >2.

5. CONCLUSION

We can summarize as follows our results: Meson fields are naturally and directly connected to the global topology of spacetime; higher-spin fields, however, have no such clear-cut relationship with the manifold's geometry due to difficulties in the construction of cohomology operators associated with Dirac's differential operator.

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Continuum S state wavefunctions for the Debye potential by Ecker–Weizel approximations[a)]

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Analytical expressions for the continuum eigenfunctions for the Debye (Yukawa) or shielded Coulomb potential are derived by using Ecker–Weizel approximations. The results are used to obtain the S matrix in closed form.

The eigenvalue problem involving the Debye or static screened coulomb potential $V(r) = -e^2\exp(-\alpha r)/r$ [α = the screening parameter] is relevant to several areas of physics.[1] Particularly, the fundamental importance of this potential in studying the properties of interacting Boltzmann particles has motivated a large number of works, both numeric and analytic in nature. The scope of the analytical approach to the problem is, however, limited because of obvious reasons. In spite of this, the analytical work of Ecker and Weizel,[2] dealing with the discrete eigenvalue problem, has recently received a considerable attention. For example, Lam and Varshni[3] have revised the expression for the Ecker–Weizel eigenvalue. The modified expression has been found to yield better energies than the original one. In the present note we shall derive analytical expressions for the eigenfunctions as well as the S matrix for the Debye Hamiltonian.

The s state radial Schrödinger equation for this potential is given by

$$\frac{1}{2}\frac{d^2}{dr^2}(rR) + \left[\frac{e^{-\delta r}}{r} - E\right](rR) = 0. \tag{1}$$

We shall use atomic units ($e = \hbar = m_e = 1$) throughout. Here $\delta = \alpha a_0$ a dimensionless screening parameter. In the units used the first Bohr radius $a_0 = 1$. The standard substitutions

$$rR = e^{-\alpha_n r} v(r) \tag{2}$$

and

$$\delta r = -\ln(1-x) \tag{3}$$

with

$$E_n = \tfrac{1}{2}\alpha_n^2 \tag{4}$$

transform Eq. (1) in the form

$$x(1-x)\frac{d^2v}{dx^2} - \beta x\frac{dv}{dx} - \frac{2}{\delta}\frac{x}{\ln(1-x)}v = 0, \tag{5}$$

where

$$\beta = 1 + 2\alpha n/\delta. \tag{6}$$

The object $(2/\delta)\cdot[x/\ln(1-x)]$ is a slowly varying function of x and to a first approximation it is a constant[2] $-\gamma$ given by

$$\gamma = \frac{2}{\delta^2}\cdot(1 - e^{-\delta\bar{r}})\bar{r}, \tag{7}$$

where $\bar{r}$ represents some sort of a mean distance of the electron in the considered quantum state. With this approximation Eq. (5) has been solved to get the expression for the eigenvalue

$$En = \frac{1}{2}\left[\frac{1}{n}\cdot\frac{1-e^{-\delta\bar{r}}}{\delta\bar{r}} - \frac{n\delta}{2}\right]^2. \tag{8}$$

To obtain the bound state eigenfunction of Eq. (5), we assume

$$v(x) = x^{\mu}(1-x)^{\lambda} f(x). \tag{9}$$

The wavefunction vanishes at $x=0$ [$r=0$]. It also vanishes at $x=1$ [$r=\infty$]. Combining Eqs. (5), (7), and (9), we get

$$[x(1-x)f''(x) + [2\mu(1-x) - \beta x - 2\lambda x]f'(x) + \left[\frac{\lambda(\lambda-1+\beta)x}{1-x} - 2\lambda\mu + \frac{\mu(\mu-1)(1-x)}{x} - \beta\mu + \gamma\right]f(x) = 0. \tag{10}$$

Here primes on f denote differentiation with respect to x. If the parameters μ and λ are chosen to be $\mu = 1$ and $\lambda = 1-\beta$, the factor of f does not depend on x and the equation reduces to the hypergeometric differential equation

$$x(1-x)f''(x) + \{2-(4-\beta)x\}f'(x) - \{2-\beta-\gamma\}f(x) = 0. \tag{11}$$

This leads to the well behaved (unnormalized) bound state solution

$$(rR) = e^{\alpha_n r}(1-e^{-\delta r})\,{}_2F_1(a,b;c;1-e^{-\delta r}), \tag{12}$$

where

$$a = \tfrac{1}{2}[3-\beta+\xi],\quad b = \tfrac{1}{2}[3-\beta-\xi],\quad c = 2 \tag{13}$$

with

$$\xi = [(1-\beta)^2 + 4\gamma]^{1/2}. \tag{14}$$

Interestingly, the power series expansion of ${}_2F_1(a,b;c;1-e^{-\delta r})$ together with $\alpha_n = -\sqrt{2E_n}$ delineates Eq. (12) in the variational form of the Yukawa wavefunction used by Hulthén and Laurikainen.[4] For subsequent discussion it will be useful to write the function (rR) as the sum of two parts. The desired result can be obtained with the help of well-known transformation

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formulas among the hypergeometric functions.[5] We thus have

$$(rR)=\frac{\Gamma(\beta-1)}{\Gamma(\frac{1}{2}(1+\beta-\xi))\Gamma(\frac{1}{2}(1+\beta+\xi))}e^{\alpha_n r}$$
$$\times{}_2F_1(\tfrac{1}{2}(1-\beta-\xi),\ \tfrac{1}{2}(1-\beta+\xi);2-\beta;e^{-\delta r})$$
$$+\frac{\Gamma(1-\beta)}{\Gamma(\frac{1}{2}(3-\beta+\xi))\Gamma(\frac{1}{2}(3-\beta-\xi))}e^{-\alpha_n r}$$
$$\times{}_2F_1(\tfrac{1}{2}(\beta+\xi-1),\ \tfrac{1}{2}(\beta-\xi+1);\beta;e^{-\delta r}). \tag{15}$$

The continuum state solution $u(r)$, regular at the origin, can be obtained from Eq. (15) through the analytic continuation $\alpha_n\to -ik$. This yields

$$u(r)=\frac{1}{2ik}\left[\frac{\Gamma(1+2ik/\delta)}{\Gamma(1+ik_1/\delta)\Gamma(1+ik_2/\delta)}e^{ikr}\right.$$
$$\times{}_2F_1\left(\frac{-ik_2}{\delta},\frac{-ik_1}{\delta};1-\frac{2ik}{\delta};e^{-\delta r}\right)$$
$$-\frac{\Gamma(1-2ik/\delta)}{\Gamma(1-ik_1/\delta)\Gamma(1-ik_2/\delta)}e^{-ikr}$$
$$\left.\times{}_2F_1\left(\frac{ik_2}{\delta},\frac{ik_1}{\delta};1+\frac{2ik}{\delta};e^{-\delta r}\right)\right], \tag{16}$$

where

$$k_1=k+(k^2-\gamma\delta^2)^{1/2},$$
$$k_2=k-(k^2-\gamma\delta^2)^{1/2}. \tag{17}$$

The two addenda on the right-hand side of Eq. (16) transform into one another under the substitution $k\rightleftharpoons -k$. Following Newton,[6] therefore, we conclude that the nearest analog of the Jost function is given by

$$D(k)=\frac{\Gamma(1+2ik/\delta)}{\Gamma(1+ik_1/\delta)\Gamma(1+ik_2/\delta)}. \tag{18}$$

Obviously, the Debye S matrix is

$$S=D(-k)/D(k). \tag{19}$$

As a usual check on the validity of the present approach we find that, in the limit $\delta\to 0$, Eqs. (8) and (18) go over to the corresponding quantities for the Coulomb potential given in Newton. This can be accomplished by noting that for the limiting case, $\gamma\delta\sim 2$, $k_1/\delta\sim 2k/\delta$ and $k_2/\delta\sim 1/k$. The results of the present paper can be used for the purpose of investigating the properties of the Debye scattering amplitude. These can also be used as starting points of perturbation calculations.

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Constraints of the Lorentz–Dirac equation

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Baylis and Huschilt pointed out that the Lorentz–Dirac equation with the usual constraint [$\ddot{x}^{\mu}(\tau)\to 0$ as $\tau\to\infty$] for physical solution, may permit two or more solutions to some problems. We impose another constraint to make the physical solution unique.

1. INTRODUCTION

The Lorentz–Dirac equation[1,2]

$$\dot{u}^{\mu}=\frac{e}{mc}F^{\mu\nu}u_{\nu}+\frac{1}{b}\left(\ddot{u}^{\mu}-\frac{1}{c^2}u^{\mu}\dot{u}^{\nu}\dot{u}_{\nu}\right) \tag{1}$$

is usually accepted to describe the classical motion of a charged particle in a force field including the radiative reaction. Here $1/b$ is equal to $2/3(e^2/mc^3)$, u^{μ} is the proper time derivative of the position x^{μ}: $\dot{x}^{\mu}=u^{\mu}$ and $\dot{u}^{\mu}=du^{\mu}/d\tau$. Equation (1) is a third-order differential equation. For given initial values of $x^{\mu}(0)$ and $\dot{x}^{\mu}(0)$, Eq. (1) has infinitely many solutions. To ensure an acceptable physical solution, one usually regards the initial acceleration $\ddot{x}^{\mu}(0)$ as a parameter and imposes the additional constraint[1]

$$\lim_{\tau\to\infty}\ddot{x}^{\mu}(\tau)=0 \tag{2}$$

to implicitly determine the initial acceleration parameter $\ddot{x}^{\mu}(0)$ and assume that one and only one such physical solution exists. In Ref. 3, Plass made rather extensive studies to determine the physical solution of Eqs. (1) and (2). He concluded that if the force field satisfies some general criteria, then there always exists a unique physical solution to Eqs. (1) and (2) for given initial values of $x^{\mu}(0)$ and $\dot{x}^{\mu}(0)$. He then argued that one should accept the equation of motion including the force of radiative reaction, Eqs. (1) and (2), as an exact equation for a charged point particle within the framework of classical theory. But recently, Baylis and Huschilt[4] pointed out that there are at least two solutions for a specific problem. The second solution of Ref. 4 exists only when the distance between the rest particle and the force field region is very small or the force field strength is very strong. Although we can argue that in the case of such a small distance or such a strong field strength, the classical description is no longer valid, we should also not that, as in the case of classical mechanics, when we consider the complete classical equation of motion, we always assume that the equation of motion is valid for all distances no matter how small and for all force field strengths no matter how strong. We further assume that the solution if it exists is unique, that is, we assume that the mathematical description of classical theory is complete up to any small distance and any strong field strength in spite of the fact that in such a case the classical description is no longer adequate to describe the actual physical situation. Thus we should reexamine the constraint condition to determine whether it is complete or not.

In Sec. 2, we use a specific problem to demonstrate the following point, which is pointed out first by Baylis and Huschilt in Ref. 4:

"The constraint condition is not complete, that is, for a given initial values of $x^{\mu}(0)$ and $\dot{x}^{\mu}(0)$, Eqs. (1) and (2) can have more than one solution."

After examining these solutions, we find out that the constraint condition can determine only the magnitude but not the direction of the initial acceleration $\ddot{x}^{\mu}(0)$. Thus we suggest in Sec. 3 one more constraint condition. The consequences of these constraints are discussed in Sec. 4.

2. SPECIFIC PROBLEM

We consider a particle of mass m and charge e moving in the electric field $\mathbf{E}(\mathbf{r})$,

$$\mathbf{E}(\mathbf{r})=\begin{cases}0, & 0\le r<r_1,\\ E_0\mathbf{r}/r, & r_1\le r\le r_2\\ 0, & r_2<r,\end{cases} \tag{3}$$

which may be regarded as the field created by an ideal spherical capacitor. In this case, Eq. (1) becomes

$$\dot{\mathbf{u}}=\frac{e}{m}\left(1+\frac{u^2}{c^2}\right)^{1/2}\mathbf{E}(\mathbf{r})+\frac{1}{b}\left\{\ddot{\mathbf{u}}-\frac{\mathbf{u}}{c^2}\left[\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}-\frac{(\mathbf{u}\cdot\dot{\mathbf{u}})^2}{(1+u^2/c^2)c^2}\right]\right\}, \tag{4}$$

where $u^2=\mathbf{u}\cdot\mathbf{u}$ and $\mathbf{u}=d\mathbf{r}(\tau)/d\tau$.

Consider the following initial conditions:

$$\mathbf{r}(0)=0,\quad \dot{\mathbf{r}}(0)=\mathbf{u}(0)=0. \tag{5}$$

The obvious physical solution of Eqs. (2) and (4) is $\mathbf{r}(\tau)=0$. But if we assume the initial acceleration $\dot{\mathbf{u}}(0)$ to be

$$\dot{\mathbf{u}}(0)=a\mathbf{b}, \tag{6}$$

where $a>0$ and $\mathbf{b}\cdot\mathbf{b}=1$, then we can integrate Eq. (4) step by step and easily see that

$$\mathbf{r}(\tau)=r(\tau)\mathbf{b}.$$

Equation (4) becomes

$$\dot{u}=\frac{e}{m}\left(1+\frac{u^2}{c^2}\right)E_0+\frac{1}{b}\left[\ddot{u}-\frac{u}{c^2}\;\frac{\dot{u}^2}{1+u^2/c^2}\right]\quad (r_1\le r\le r_2). \tag{7}$$

This nonlinear differential equation can be solved by

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introducing the new dependent variable $\omega(\tau)$ define by the equation

$$u(\tau) = c\sinh[\omega(\tau)/c].$$

It can be proved[4] that if the distance r_1 is sufficiently small or the field strength E_0 is sufficiently large, Eqs. (7) and (2) give us a physical solution with the initial acceleration $a > 0$. The unit vector b can be any unit vector; this means that, for this specific problem, we have infinitely many physical solutions so long as r_1 is sufficiently small or E_0 is sufficiently large.

Now let us consider the following initial conditions:

$$\mathbf{r}(0) = 0, \quad \mathbf{u}(0) = u_0\mathbf{e}_1, \tag{8}$$

where $u_0 > 0$ and $\mathbf{e}_1$ is a unit vector. If we assume the initial acceleration $\dot{\mathbf{u}}(0)$ to be

$$\dot{\mathbf{u}}(0) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2,$$

where $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ and $\mathbf{e}_2 \cdot \mathbf{e}_2 = 1$, then the motion of the charged particle will be restricted in the $(\mathbf{e}_1, \mathbf{e}_2)$ plane. If u_0 is sufficiently small, we have good reason to believe that Eqs. (7) and (2) have solution for a wide range of $\dot{\mathbf{u}}(0)$. For the special case $a_2 = 0$, it can be proved that for sufficiently small u_0, Eqs. (7) and (2) have two solutions, one with $a_1 > 0$ and the other with $a_1 < 0$.

3. ADDITIONAL CONSTRAINT

From the analysis of Sec. 2, we see the following facts:

(a) In general, the LD equation, Eq. (1) restricted by the condition, Eq. (2), has more than one physical solution for given initial position and velocity.

(b) Among the physical solutions for given initial position and velocity, every solution corresponds to a direction of the initial acceleration $\dot{\mathbf{u}}(0)$.

The constraint condition, Eq. (2), seems to determine only the magnitude but not the direction of the initial acceleration.

In order to make the LD equation a complete mathematical description, we can assume one more constraint to the LD equation. We suggest that the following additional constraint should be added to the LD equation:

"The direction of the acceleration $\dot{\mathbf{u}}(\tau_0)$ at any proper time τ_0 should be in the direction of $\dot{\mathbf{u}}_\infty(\tau_0')$, that is $\dot{\mathbf{u}}(\tau_0) = \alpha\dot{\mathbf{u}}_\infty(\tau_0')$ with $\alpha > 0$, where $\mathbf{u}_\infty(\tau)$ is the classical solution without radiative reaction, i.e., solution corresponding to the case $b \to \infty$ with initial conditions $\mathbf{r}_\infty(\tau_0) = \mathbf{r}(\tau_0)$ and $\mathbf{u}_\infty(\tau_0) = \mathbf{u}(\tau_0)$ and $\tau_0' = \min\{\tau \mid \tau \geq \tau_0, \dot{\mathbf{u}}_\infty(\tau) \neq 0\}$. If $\tau_0' = \infty$, then $\dot{\mathbf{u}}(\tau_0)$ should be zero." (9)

This additional constraint is very reasonable, for (1) it is a covariant constraint, (2) it is sufficient to isolate one solution for the specific problem discussed in Sec. 2, and (3) the particle only begins accelerating noticeably over a time interval of the order b^{-1} *before* the force is applied and b^{-1} is very small.

Now let us consider the specific problem discussed in Sec. 2. After introducing the additional constraint, Eq. (9), the problem becomes to solve the differential equation, Eq. (4), with constraint conditions, Eqs. (2) and (9).

Consider the case of the initial conditions, Eq. (5), then the exact solution for the classical motion without radiative reaction is

$$\mathbf{r}_\infty(\tau) = 0, \quad \mathbf{u}_\infty(\tau) = 0, \quad \dot{\mathbf{u}}_\infty(\tau) = 0.$$

Thus the initial acceleration $\dot{\mathbf{u}}(0)$ should be zero according to the constraint condition, Eq. (9) and the unique physical solution is

$$\mathbf{r}(\tau) = 0, \quad \mathbf{u}(\tau) = 0, \quad \dot{\mathbf{u}}(\tau) = 0.$$

The other solutions discussed in Sec. 2 become nonphysical solutions. We will come back to this point later.

Consider the case of initial conditions, Eq. (8): the classical solution without radiative reaction can be easily obtained and

$$\dot{\mathbf{u}}_\infty(\tau_0') = \frac{c}{m}\left(1 + \frac{u_0^2}{c^2}\right)^{1/2} E_0\mathbf{e}_1.$$

Thus the initial acceleration $\dot{\mathbf{u}}(0)$ should be

$$\dot{\mathbf{u}}(0) = a_1\mathbf{e}_1, \quad a_1 > 0.$$

The problem becomes a one-dimensional problem. It can be proved that only one solution exists.

4. DISCUSSION

(a) For a given initial velocity and initial position, LD equation, Eq. (1), with constraint conditions Eqs. Eq. (2), does not give us a unique physical solution in general. The constraint condition, Eq. (2), seems to determine only the magnitude but not the direction of the initial acceleration.

(b) If we impose another constraint condition, Eq. (9), which can be used to determine the direction of the initial acceleration, we find out that it is sufficient to isolate one solution among the many physical solutions in the problem discussed in Sec. 2.

(c) The additional constraint, Eq. (9), is a covariant constraint.

(d) It is still not an easy problem to prove that the LD equation, Eq. (1) with constraint conditions Eqs. (2) and (9), does have unique solution for general force field.

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The asymptotic behavior of bound eigenfunctions of Hamiltonians for single-variable systems

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An extremely simple technique for examining the asymptotic behavior of bound eigenfunctions of Hamiltonians for single-variable systems is presented. A few simple examples are studied.

INTRODUCTION

During the last five years much work has been done in determining the asymptotic behavior of bound eigenfunctions.[1-9] The specific case of a three-particle system with asymptotically vanishing potentials was studied earlier by Slaggie and Wichmann.[10] Very recently discovered evidence strongly suggests that an optimal bound should satisfy the differential equation asymptotically,[6-9] as does Slaggie and Wichmann's bound. We shall demonstrate this fact for a reasonably well-behaved single-variable system. (A precise definition of "well-behaved" will be given later.)

Theorem: Let $W(x)$ be a positive differentiable weight function defined on (a,∞), where a can be either finite or $-\infty$. Denote $W(x)\,(V(x)-E)$ by $F(x)$. Suppose that $f(x)$ and $g(x)$ are solutions of

$$\frac{d}{dx}\left(W(x)\,\frac{df}{dx}\right)=F(x)f \tag{1}$$

and

$$\frac{d}{dx}\left(W(x)\,\frac{dg}{dx}\right)=G(x)g, \tag{2}$$

respectively, where f, g, f', and g' are all in $L^2((a,\infty),\ W(x)dx)$. If there exists x_0 such that $F(x)\geqslant G(x)$ and $g(x)>0$ for all $x\geqslant x_0$, there exists $x_0'\geqslant x_0$ such that

$$|f(x)/f(x_0')|\leqslant g(x)/g(x_0') \tag{3}$$

for all $x\geqslant x_0'$. If $g'(x)<0$ for all $x\geqslant x_0'$, $f'(x)<0$ for all $x\geqslant x_0'$.

Proof: By Sturm's fundamental theorem,[11] $f\neq 0$ for $x\geqslant x_0'$ for some $x_0'\geqslant x_0$, so we can take f to be positive for $x\geqslant x_0'$. We multiply Eq. (1) by g and Eq. (2) by f and take the difference between the new expressions to find that

$$\frac{d}{dx}\,(W(x)(gf'-fg'))=(F(x)-G(x))fg. \tag{4}$$

We integrate from x_1 to x, where $x_1\geqslant x_0'$, to yield

$$[W(t)(gf'-fg')]_{x_1}^{x}=\int_{x_1}^{x}(F(t)-G(t))f(t)g(t)dt. \tag{5}$$

Since the integrand is nonnegative, if there exists a number $x_2\geqslant x_0$ such that $W(x_2)(g(x_2)f'(x_2)-f(x_2)g'(x_2))=b>0$, then $W(gf'-fg')\geqslant b$ for all $x\geqslant x_2$. However, then

$$\int_{x_2}^{\infty}(g(x)f'(x)-f(x)g'(x))W(x)dx=+\infty, \tag{6}$$

which contradicts the hypothesis that f, g, f', and g' are in $L^2((a,\infty),\ W(x)dx)$. Thus for all $x\geqslant x_0'$, $gf'-fg'\leqslant 0$, so $f'/f\leqslant g'/g$. Integration yields (3). Furthermore, since $gf'-fg'\leqslant 0$, $f'<0$ if $g'<0$ for all $x\geqslant x_0'$. QED

Similarly, (1) can be subtracted from (2) to yield $(W(g'-f'))'=Gg-Ff$, which in turn can be multiplied by $(g-f)$ to yield $(g-f)(W(g'-f'))'=(g-f)(Gg-Ff)$. Since $(g-f)(W(g'-f'))'=((g-f)W(g'-f'))'-W(g'-f')^2$, $((g-f)W(g'-f'))'\geqslant(g-f)(Gg-Ff)$. Hence

$$[(g(t)-f(t))W(t)(g'(t)-f'(t))]_{x_1}^{x}\geqslant\int_{x_1}^{x}(g(t)-f(t))(G(t)g(t)-F(t)f(t))\,dt. \tag{7}$$

If it is assumed that each factor in the integrand is positive, it follows that $g'-f'\leqslant 0$ under the hypothesis of the theorem *supra*. If $g'\leqslant 0$, $0\leqslant|f'|\leqslant|g'|$. The case $x\to-\infty$ can be treated analogously.

Hartman and Wintner proved a similar theorem with no restrictions on f' and g' in which it was supposed that $W=1$.[12,13] The restriction on W is unnecessary, for by letting $f(x)=z(x)g(x)$ we obtain

$$Wgz''+(2Wg'+W'g)z'+(G-F)gz=0, \tag{8}$$

to which Theorem 2.46 of Swanson's book is applicable. The method presented *supra* is more direct than the earlier techniques, and it also yields information relating f' and g'. The bounds on the derivatives are better than some previously discovered ones.[14-17] These results are more general for radial Schrödinger equations than those found recently by Bardos and Mérigot.[18]

If $F(x)$ is approximated asymptotically by functions $G_\theta(x)$, where asymptotically $F/G_\theta\lesseqgtr 1$ if $\theta\gtreqless 1$ [i.e., for any $\epsilon>0$ there exists θ such that $|1-F(x)/G_\theta(x)|<\epsilon$ for all x greater than some x_θ], then the solutions g_θ will provide asymptotic upper and lower bounds for f for $\theta<1$ and $\theta>1$, respectively. For a reasonably well-behaved F, the asymptotic behavior of f is usually considered to be given by $\lim_{\theta\to 1} g_\theta$.

On a less abstract level, suppose that there exists a constant x_0 such that $V(x)\geqslant E$ if $x\geqslant x_0$. Also suppose that $(W(V-E)^{1/2})'$ is asymptotically negligible compared with $W(V-E)$, which is true for almost all physically interesting systems. (This condition is the "reason-
$F(x)=W(x)(V(x)-E)$ and $G_\theta(x)=\theta^2W(x)(V(x)-E)-\theta(W(x)(V(x)-E)^{1/2})'$. The conditions outlined in the previous paragraph are satisfied, so since $\exp[-\theta\int_{x_0}^{x}(V(t)-E)^{1/2}dt]$ is a solution of $(Wg_\theta')'=G_\theta(x)g_\theta$, we have asymptotically for any eigenfunction ψ and for

any $\epsilon > 0$

$$\exp[-(1+\epsilon)\int_{x_\theta}^{x}(V(t)-E)^{1/2}dt] \leq |\psi(x)/\psi(x_\theta)| \leq \exp[-(1-\epsilon)\int_{x_\theta}^{x}(V(t)-E)^{1/2}dt]. \quad (9)$$

Under these circumstances we feel justified in saying that the "optimal bound" is given by $\lim_{\theta\to 1} g_\theta = g_1$, and it is clear that g_1 satisfies the differential equation asymptotically.

SOME APPLICATIONS

This section is by no means intended to be complete. It should merely illustrate the facility of using this theorem to study the asymptotic behavior of bound eigenfunctions.

This theorem allows one to derive the type of bounds which Simon discusses for eigenfunctions of Hamiltonians whose potentials are $0(|x|^{2\alpha})$.[6] If $V(x) \geq k|x|^{2\alpha} - c(k>0, \alpha>0)$, for any $0<\theta<1$ there exists x_θ such that

$$\theta^2 k|x|^{2\alpha} - \theta\sqrt{k}\,\alpha|x|^{\alpha-1} = G_\theta(x) \leq V(x) - E \quad (10)$$

for all $x \geq x_\theta$. Since $\exp[-\theta\sqrt{k}|x|^{\alpha+1}/(\alpha+1)]$ is a positive solution of $g''(x) = G_\theta(x)g(x)$, it follows that the eigenfunctions of the above Hamiltonian in one dimension are bounded asymptotically by a constant times this function. If $\alpha > 1$, $G_1(x) \leq V(x) - E$ asymptotically, so in this case the eigenfunctions are bounded asymptotically by a constant times $\exp[-\sqrt{k}|x|^{\alpha+1}/(\alpha+1)]$. Since in one dimension, functions in the domain of the momentum operator are bounded,[19] these bounds can be extended to all x. Furthermore, if $V(x) \leq k|x|^{2\alpha} + c$, the reverse inequalities hold asymptotically for $\theta > 1$ since the nth eigenfunction has at most $n-1$ nodes.[20] These results strengthen and generalize Simon's theorems for one-dimensional systems.

The next example is a central potential V in three dimensions such that V is in $L^2 + L^\infty$ and $\lim_{r\to\infty} V(r) = 0$. For this potential the eigenfunctions of the Hamiltonian are bounded.[21] For total angular momentum $\sqrt{l(l+1)}$, an eigenfunction ψ satisfies

$$r^{-2}((r^2\psi')' - l(l+1)\psi) = (V(r) - E)\psi. \quad (11)$$

Let $\psi = r^l\Phi$. Since

$$r^{-2}((r^2\psi')' - l(l+1)\psi) = r^l(r^{-2l-2}(r^{2l+2}\Phi')'), \quad (12)$$

satisfies $(W\Phi')' = F(r)\Phi$ with $W = r^{2l+2}$ and $F = W(V(r) - E)$. Assuming that $E < 0$ and letting

$$G_\theta(r) = -\theta^2 E r^{2l+2} - (2l+2)\theta\sqrt{-E}\, r^{2l+1}, \quad (13)$$

we see that for any $0<\theta<1$ there exists r_θ such that $G_\theta(r) \leq F(r)$ for all $r \geq r_\theta$. Since $\exp(-\theta\sqrt{-E}\, r)$ is a solution of $(Wg')' = G_\theta(r)g$, the eigenfunctions of this Hamiltonian are bounded by $K_\theta r^l \exp(-\theta\sqrt{-E}\, r)$ for all r. The factor of r^l can be removed by choosing a new θ' such that $\theta < \theta' < 1$.

In the example *supra*, suppose $V(r)$ can be bounded from above or from below by a homogeneous function of r. We shall study the implications of these asymptotic bounds on the potential. It will always be assumed that the constant k is positive.

(Ia): $V(r) \leq kr^{-1+\epsilon}$ asymptotically, with $0<\epsilon<1$. We let

$$g(r) = r^{-1}\exp[-\int_{r_0}^{r}(kt^{-1+\epsilon} - E)^{1/2}dt, \quad (14)$$

so

$$r^{-1}(rg)'' = \left\{-E + kr^{-1+\epsilon} + \frac{k}{2}(1-\epsilon)\frac{r^{-2+\epsilon}}{(kr^{-1+\epsilon}-E)^{1/2}}\right\}g. \quad (15)$$

Eventually the last term will overwhelm any centrifugal potential, so we conclude that there exists r_0 such that for all $r \geq r_0$

$$\left|\frac{\psi(r)}{\psi(r_0)}\right| \geq \frac{r_0}{r}\exp[-\int_{r_0}^{r}(kt^{-1+\epsilon} - E)^{1/2}dt]. \quad (16)$$

(Ib): $V(r) \geq kr^{-1+\epsilon}$ asymptotically, with $0<\epsilon<1$. We let

$$g(r) = r^{-1}(kr^{-1+\epsilon} - E)^{-1/4}\exp[-\int_{r_0}^{r}(kt^{-1+\epsilon} - E)^{1/2}dt], \quad (17)$$

so

$$r^{-1}(rg)'' = \left\{-E + kr^{-1+\epsilon} + r^{-2}\left[\left(\frac{k}{4}(1-\epsilon)\frac{r^{-1+\epsilon}}{kr^{-1+\epsilon}-E}\right)^2 + \frac{k}{4}(1-\epsilon)\left(\frac{(-2+\epsilon)r^{-1+\epsilon}}{kr^{-1+\epsilon}-E} + \frac{k(1-\epsilon)r^{-2+2\epsilon}}{(kr^{-1+\epsilon}-E)^2}\right)\right]\right\}g. \quad (18)$$

The last term is asymptotically negative, so it eventually will "underwhelm" any centrifugal potential, so we can infer that there exists r_0 such that for all $r \geq r_0$

$$\left|\frac{\psi(r)}{\psi(r_0)}\right| \leq \frac{K(r_0)}{r}\exp(-\int_{r_0}^{r}(kt^{-1+\epsilon} - E)^{1/2}dt). \quad (19)$$

In (I) the integral inside the exponential can be removed by expanding the integrand as a Taylor series in $t^{-1+\epsilon}$ and then integrating this series term-by-term. Only a finite number of terms need to be kept, for the remainder tends to 0 as r becomes infinite. It will be noted that these bounds have no dependence on the angular momentum channel.

(IIa): $V(r) \leq kr^{-1}$ asymptotically. We let

$$g(r) = r^{-1}r^{-\alpha}(1+\beta r^{-1})\exp(-\sqrt{-E}\,r), \quad (20)$$

where $\alpha = k/(2\sqrt{-E})$. It is straightforward to verify that

$$r^{-1}(rg)'' = \left\{-E + \frac{k}{r} + \frac{1}{r^2}\,\frac{1}{1+\beta r^{-1}}[(2\sqrt{-E}\,\beta + (\alpha+1)(\alpha+\beta(\alpha+2)r^{-1})]\right\}g. \quad (21)$$

By choosing β sufficiently large it is possible to make the last term overwhelm any centrifugal potential, so we conclude that asymptotically an eigenfunction ψ of $-\nabla^2 + V$ is bounded from below by a constant multiple of

$$r^{k/(2\sqrt{-E})-1}\exp(-\sqrt{-E}\,r). \quad (22)$$

(IIb): $V(r) \geq kr^{-1}$ asymptotically. This case can be treated by letting $\epsilon = 0$ in (Ib). We infer the existence of an asymptotic upper bound of the form (22).

(IIIa): $V(r) \leq kr^{-1-\epsilon}$ with $\epsilon > 0$. We examine

$$g(r)=r^{-1}(1+\beta r^{-\alpha})\exp(-\sqrt{-E}r), \tag{23}$$

with $\alpha=\min(1,\epsilon)$. Since

$$r^{-1}(rg)''=\left\{-E+\frac{\beta\alpha r^{-\alpha-1}}{1+\beta r^{-\alpha}}\,(2\sqrt{-E}+(\alpha+1)r^{-1})\right\}g, \tag{24}$$

we can always choose β such that the right side of (24) exceeds $V(r)+r^{-2}l(l+1)-E$. Hence there exists an asymptotic lower bound of the form

$$r^{-1}\exp(-\sqrt{-E}\,r) \tag{25}$$

to an eigenfunction of $-\nabla^2+V$.

(IIIb): $V(r)\geq -kr^{-1-\epsilon}$ with $\epsilon>0$. We examine

$$g(r)=r^{-1}(1-\beta r^{-\epsilon})\exp(-\sqrt{-E}r). \tag{26}$$

Since

$$r^{-1}(rg)''=\left\{-E-\frac{\beta\epsilon r^{-\epsilon-1}}{1-\beta r^{-\epsilon}}\,(2\sqrt{-E}+(\epsilon+1)r^{-1})\right\}g, \tag{27}$$

we choose β so large that asymptotically the right side of (27) does not exceed $V(r)-E\leq V(r)+r^{-2}l(l+1)-E$. Hence an eigenfunction of $-\nabla^2+V$ has an upper bound of the form (25). For potentials which are central and tend smoothly to 0 as r becomes infinite, this result is a generalization of one of de Alfaro and Regge.[22]

(IVa): $V(r)\leq -kr^{-1}$ asymptotically. We let

$$g(r)=r^{-1}r^{\alpha}(1+\beta r^{-1})\exp(-\sqrt{-E}r), \tag{28}$$

where $\alpha=-k/(2\sqrt{-E})$. We obtain the analog of (21):

$$r^{-1}(rg)''=\left\{-E-\frac{k}{r}+\frac{1}{r^2}\,\frac{1}{1+\beta r^{-1}}\,[2\sqrt{-E}\beta + (\alpha-1)(\alpha+\beta(\alpha-2))r^{-1})]\right\}g. \tag{29}$$

By choosing β sufficiently large we can make the last term overwhelm any centrifugal potential, so asymptotically an eigenfuncation ψ of $-\nabla^2+V$ has a lower bound of the form

$$r^{k/(2\sqrt{-E})-1}\exp(-\sqrt{-E}r). \tag{30}$$

(IVb): $V(r)\geq -kr^{-1}$ asymptotically. This case is treated by setting $\epsilon=0$ in (Vb) *infra*. We obtain an upper bound of the form (30).

(Va): $V(r)\leq -kr^{-1+\epsilon}$ asymptotically, with $0<\epsilon<1$. Let

$$g(r)=r^{-1}(-kr^{-1+\epsilon}-E)^{-1/2}\exp\left[-\int_{r_0}^{r}(-kt^{-1+\epsilon}-E)^{1/2}dt\right], \tag{31}$$

so

$$r^{-1}(rg)''=\left\{-E-kr^{-1+\epsilon}+\frac{k}{2}\,\frac{(1-\epsilon)r^{-2+\epsilon}}{(-kr^{-1+\epsilon}-E)^{1/2}} + r^{-2}\left[\left(\frac{k}{2}\,\frac{(1-\epsilon)r^{-1+\epsilon}}{-kr^{-1+\epsilon}-E}\right)^2-\frac{k}{2}\,(1-\epsilon) \times\left(\frac{(-2+\epsilon)r^{-1+\epsilon}}{-kr^{-1+\epsilon}-E}-\frac{k(1-\epsilon)r^{-2+2\epsilon}}{(-kr^{-1+\epsilon}-E)^2}\right)\right]\right\}g. \tag{32}$$

The last term asymptotically overwhelms any centrifugal potential, so we infer that there exist r_0 and $K(r_0)$ such that for all $r\geq r_0$,

$$\left|\frac{\psi(r)}{\psi(r_0)}\right|\geq\frac{K(r_0)}{r}\exp\left[-\int_{r_0}^{r}(-kt^{-1+\epsilon}-E)^{1/2}dt\right]. \tag{33}$$

(Vb): $V(r)\geq -kr^{-1+\epsilon}$ asymptotically, with $0<\epsilon<1$. As in (Ia), we let

$$g(r)=r^{-1}\exp\left(-\int_{r_0}^{r}(-kt^{-1+\epsilon}-E)^{1/2}dt\right), \tag{34}$$

so

$$r^{-1}(rg)''=\left\{-E-kr^{-1+\epsilon}-\frac{k}{2}\,(1-\epsilon)\,\frac{r^{-2+\epsilon}}{(-kr^{-1+\epsilon}-E)^{1/2}}\right\}g. \tag{35}$$

The last term is negative, so it is less than any centrifugal potential, so we conclude an eigenfunction of $-\nabla^2+V$ satisfies

$$\left|\frac{\psi(r)}{\psi(r_0)}\right|\leq\frac{r_0}{r}\exp\left[-\int_{r_0}^{r}(-kt^{-1+\epsilon}-E)^{1/2}dt\right], \tag{36}$$

for all r exceeding some r_0.

Thus we have completely examined all cases in which the potential $V(r)$ is asymptotically bounded above or below by a function homogeneous in r of negative degree. We have seen that there is no dependence of the gross asymptotic behavior on the angular momentum. Although the results of this section almost certainly are fairly old, the author does not know of a comprehensive presentation of them.[23] It has been seen that the method outlined in the Introduction easily yields very sharp results for the asymptotic behavior of bound eigenfunctions of Sturm–Liouville operators.

Of course, we could use (7) to study the asymptotic behavior of the derivatives of the eigenfunctions in these examples, but presumably the technique is sufficiently straightforward that doing so explicitly here is not necessary.

Finally, we would like to present a general result on the effect of a positive tail of a potential. Suppose $V=V_1+V_+$, where $V_+\geq 0$, and $-\nabla^2+V$ and $-\nabla^2+V_1$ are self-adjoint on the domain of $-i\nabla$. Assume that $-\nabla^2+V$ has an eigenfunction ψ of energy E, where ψ has a finite number of nodes. If one can find a potential V_2 such that $V_2(r)=V_1(r)$ for all r exceeding some r_0 and $-\nabla^2+V_2$ has an eigenfunction Φ of energy E, then asymptotically Φ provides an upper bound to ψ. Normally one would find V_2 by adding to V_1 a sufficiently positive function with support in $r\leq r_0$ for a sufficiently large r_0.

CONCLUSIONS

The "direct comparison" method outlined in this article appears to be very useful for studying the asymptotic behavior of self-adjoint second-order ordinary differential systems. We see here a reflection of the physical intuition that the more positive the potential becomes asymptotically, the more quickly the eigenfunctions decrease asymptotically.

It is worth remarking that the applicability of this method is quite insensitive to the short-range behavior of the potential provided that the eigenfunctions are in the domain of the momentum operator, which is a physically motivated restriction. This situation is in accord with our physical intuition, for we would not expect the short-range nature of the potential to influence the long-range behavior of the solutions.

This theorem does not readily generalized to higher dimensions. The main obstacle is that the left side of (5) is replaced with a surface integral of $g\nabla f - f\nabla g$, and the components of this vector need not be monotonic.

The bounds discussed in Eq. (9) are very similar to the behavior of the wavefunction in the WKB approximation. However, the assumptions made in our treatment are quite different, and indeed are completely rigorous.

The theorem can be extended to the case of a finite interval provided that the product $W\psi\Phi'$, vanishes at the boundary for all ψ, Φ in the domain of the momentum operator.

It should be mentioned that similar "Sturmian" techniques have been used very recently to study the short-range behavior of "charmonium" wavefunctions.[24,25]

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Projected basis set for the irreducible representation $\{2^{N/2-S}, 1^{2S}\}$ of U(n)

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The transformation properties of a projected basis for the irreducible representation (IR), $\{2^{N/2-S}, 1^{2S}\}$, of U(n) under the elementary generators of the group have been studied. It has been found that these transformations are identical (to within a phase factor) with those of the standard bases spanning the given IR. The correspondence between this basis set and the standard basis set has also been indicated.

1. INTRODUCTION

The realization of an orthonormal basis spanning the irreducible representation (IR), $\{2^{N/2-S}, 1^{2S}\}$, of the unitary group U(n) is of basic importance in the study of many-electron systems. The fact that the permutations belonging to the symmetric group, S_N, on the electron coordinates commute with the generators, E_{ij}, of U(n) enable one to generate such a basis using the idempotents $\{\omega^{\lambda}_{rs} \mid r, s = 1, \ldots, f^{\lambda}_n; [\lambda] = [2^{N/2-S}, 1^{2S}]\}$ defining the algebra of S_N.[1-3] In a recent series of notes, Harter and Patterson[4,5] carried out a detailed study of this problem and generated the canonical Weyl basis. Since these authors[4,5] defined only the matrix elements of the elementary generators, E_{ii+1} (E_{i+1i}) over the Weyl basis through a set of "rules," the handling of general E_{ij} became quite complicated. Following this work,[4,5] Sarma and Rettrup[6] used a slightly different choice for the reducible tensor basis of U(n) to obtain the Young projections spanning the IR $\{2^{N/2-S}, 1^{2S}\}$ of U(n). A computer program was also developed to obtain the matrix elements of general E_{ij} over this basis.[7]

The characteristic feature of the Weyl[8,9] or the Gel'fand[10-12] basis spanning the given IR of U(n) is that E_{ii+1} (E_{i+1i}) transform each of these basis functions into at most two others.[4,13,14] In this note we have attempted a detailed analysis of the projected basis[6] under E_{ii+1}(E_{i+1i}). We have shown that the transformations induced in this basis are similar to that of Weyl or the Gel'fand basis for $\{2^{N/2-S}, 1^{2S}\}$ of U(n). The transformation properties have been studied in Sec. 2, and a brief discussion is presented in Sec. 3.

2. THE PROJECTED BASES AND THEIR TRANSFORMATION PROPERTIES

Let $\{|\phi^{\mu}_i\rangle \; \mu = 1, N; i = 1, \ldots, n\}$ form a basis for the fundamental representation of U(n). The direct product,

$$|\phi_{(i_p)}\rangle = |\phi^1_{i_1}\phi^2_{i_1}\cdots\phi^{2p-1}_{i_p}\phi^{2p}_{i_p}\phi^{2p+1}_{i_{p+1}}\cdots\phi^N_{i_{N-p}}\rangle \quad (1)$$

$$i_1 < i_2 < \cdots < i_p, \quad i_{p+1} < i_{p+2} < \cdots < i_{N-p}),$$

with $[0, N-n]_{\max} \leq p \leq N/2 - S$, provides a reducible tensor representation for the given IR of U(n). Using the projection operators[1,4]

$$\omega^{\lambda}_{rs} = (f^{\lambda}_N / N!)^{1/2} \sum_{P \in S_N} [P]^{\lambda}_{rs} P, \quad (2)$$

for the IR $[\lambda] \equiv [2^{N/2-S}, 1^{2S}]$ of S_N, we find that the nonzero projections,

$$|\phi_{(i_p)}; rs\rangle^{\lambda} = \omega^{\lambda}_{rs} |\phi_{(i_p)}\rangle, \quad (3)$$

define a basis set spanning the given IR of U(n).[6] The transformation induced by a generator, E_{ij}, of U(n) on the basis set defined by Eq. (3), can then be determined as[6]

$$E_{ij}|\phi_{(i_p)}; rs\rangle^{\lambda} = \sum_{s'} [P_{(r'_i r'_{i\pm1} \cdots r_{j\mp1} r_j)}]^{\lambda}_{s's} \times |\phi_{(i'_p)}; rs'\rangle^{\lambda}, \quad (4)$$

$$i \notin \{i_k \mid k = 1, p_2, p+1, \ldots, N-p\} \; i_{p+1} \leq i_{N-p}),$$

$$E_{ij}|\phi_{(i_p)}; rs\rangle^{\lambda} = -\sum_{s'} [P_{(r_i r_{i\pm1} \cdots r'_{j\mp1} r'_j)}]^{\lambda}_{s's} \times |\phi_{(i'_p)}; rs'\rangle^{\lambda}, \quad (5)$$

$$i_1 \leq j \leq i_p, \quad i_{p+1} \leq i \leq i_{N-p},$$

$$E_{ij}|\phi_{(i_p)}; rs\rangle^{\lambda} = \sqrt{2} \sum_{s'} [P_{(r_2 r_3 \cdots \bar{r}_i)(r_1 r_2 \cdots r_j)}]^{\lambda}_{s's} \times |\phi_{(i^0_{(p+1)})}; rs'\rangle^{\lambda}, \quad (6)$$

$$i_{p+1} \leq i, j \leq i_{N-p}.$$

Restricting the above transformation to those induced E_{ii+1}, it is worth verifying whether this projected basis also yields the same transformations as the standard basis of U(n).[4,13,14] For this purpose consider the projection resulting from a completely ordered tensor product,

$$|\phi_{(p)}; rs\rangle^{\lambda} = \omega^{\lambda}_{rs} |\phi^1_1\phi^2_1 \cdots \phi^{2p-1}_p\phi^{2p}_p\phi^{2p+1}_{p+1} \cdots \phi^N_{N-p}\rangle. \quad (7)$$

If, further, in Eq. (7) $p = N/2 - S$, the only nonzero projection which results for a fixed value of the index r is $|\phi_{(N/2-S)}; r1\rangle^{\lambda}$ in the Yamanouchi ordering.[15] For such a state we can readily verify that all the raising generators, E_{ii+1}, of U(n) yield zero transformations implying that it is the highest weight state (HWS) of the basis set. For an arbitrary state, $|\phi_{(p)}; rs\rangle^{\lambda}$, consider the effect of E_{p+1p}. We have, using Eq. (5) and the fact that the matching permutation in

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this case is the identity,

$$E_{p+1\ p}|\phi_{(p)};rs\rangle^\lambda=-|\phi'_{(p)};rs\rangle^\lambda, \tag{8}$$

where

$$|\phi'_{(p)};rs\rangle^\lambda = \omega^\lambda_{rs}|\phi^1_1\phi^2_1\cdots\phi^{2p-1}_{p+1}\phi^{2p}_{p+1}\phi^{2p+1}_p\cdots\phi^N_{N-p}\rangle. \tag{9}$$

Thus, $E_{p+1\ p}$ does not change the standard tableau and it replaces a doubly occupied orbital of $|\phi_{(p)}\rangle$ by a singly occupied one. This result is similar to the one given by Harter and Patterson[4] except for a phase factor.

Consider now $E_{N-p+1\ N-p}|\phi_{(p)};rs\rangle^\lambda$ where ϕ_{N-p+1} is an unoccupied orbital for $|\phi_{(p)}\rangle$. Using Eq. (4) and the fact that again the matching permutation is the identity, we have,

$$E_{N-p+1\ N-p}|\phi_{(p)};rs\rangle^\lambda=|\phi''_{(p)};rs\rangle^\lambda, \tag{10}$$

where

$$|\phi''_{(p)};rs\rangle^\lambda = \omega^\lambda_{rs}|\phi^1_1\phi^2_1\cdots\phi^{2p-1}_p\phi^{2p}_p\phi^{2p+1}_{p+1}\cdots\phi^N_{N-p+1}\rangle^\lambda. \tag{11}$$

This result is again similar to the one given by Harter and Patterson.[4]

We now consider the rest of the elementary generators E_{i-1i} $(i=p+2,\ldots,N-p)$ which yield nonzero transformations on $|\phi_{(p)};rs\rangle^\lambda$. Each of these operations leads to a contraction $(\phi_{i-1},\phi_i)\to(\phi_{i-1},\phi_{i-1})$ and according to Eq. (6) involves only the permutations of the electron coordinates of singly occupied orbitals of $|\phi_{(p)}\rangle$. Hence, without any loss of generality, we label the particle coordinates of these orbitals as $1,2,\ldots,N-2p$ in place of $p+1,\ldots,N$. The same considerations also imply that any change in the tableau, s, due to the permutations of these coordinates, leads to others with relabeling in this portion of the tableau only. In view of these considerations we introduce a more convenient tableau notation.

$$\mathcal{S}\equiv\begin{array}{|c|c|}\hline 1 & q_1\\ \hline 2 & q_2\\ \hline \cdots & \cdots\\ \hline K & q_k\\ \hline \vdots & \\ \hline\end{array}\equiv(q_1,q_2,\cdots,q_k). \tag{12}$$

Using this definition, the projected state of Eq. (7) is represented as

$$|\phi_{(p)};rs\rangle^\lambda = \omega^\lambda_{r(q_1,\ldots,q_k)}|\phi_0\phi^1_{p+1}\phi^2_{p+2}\cdots\phi^{q_i-1}_{p+q_i-1}\phi^{q_i}_{p+q_i}\cdots\phi^{N-2p}_{N-p}\rangle, \tag{13}$$

where ϕ_0 represents the doubly occupied portion of $\phi_{(p)}$.

Consider, first, the effect of the raising generator, $E_{p+q_1-1\ p+q_1}$ on the state $|\phi_{(p)};rs\rangle^\lambda$ of Eq. (13). On using Eq. (6) we have,

$$E_{p+q_1-1\ p+q_1}|\phi_{(p)};rs\rangle^\lambda = \sqrt{2}\sum_{s'}[P^{(q_1)}]^\lambda_{s's}|\phi'_{(p+1)};rs'\rangle^\lambda, \tag{14}$$

where

$$P^{(q_1)}=P^{(q_1-1)}(q_1-1,q_1)(q_1-2,q_1-1) \tag{15a}$$

$$\text{subject to } P^{(2)}=e, \tag{15b}$$

and

$$|\phi'_{(p+1)}\rangle=|\phi_0\phi^1_{p+q_1-1}\phi^2_{p+q_1-1}\phi^3_{p+1}\cdots\phi^{N-2p}_{N-p}\rangle. \tag{16}$$

The permutation $P^{(q_1)}\in S_{q_1}$ is over the first q_1 particles defining the singly occupied portion of $|\phi_{(p)}\rangle$. Further, since $|\phi'_{(p+1)}\rangle$ is symmetric in the first two particles, $|\phi'_{(p+1)};rs'\rangle^\lambda=0$ unless $s'=(2,q_2,\ldots,q_k)$. The use of these results in Eq. (14) along with the behavior of the IR's of S_{q_1} under elementary transpositions leads to

$$\begin{aligned}E_{p+q_1-1\ p+q_1}&|\phi_{(p)};rs\rangle^\lambda\\ &=\sqrt{2}\,[P^{(q_1)}]^\lambda_{(2,q_2,\ldots,q_k)(q_1,\ldots,q_k)}\times|\phi'_{(p+1)};r(2q_2,\ldots,q_k)\rangle^\lambda\\ &=-\sqrt{2}\left(\frac{1}{q_1-1}[P^{(q_1-1)}]^\lambda_{(2,q_2,\ldots,q_k)(q_1,q_2,\ldots,q_k)}+\frac{\sqrt{(q_1-2)q_1}}{q_1-1}[P^{(q_1-1)}]^\lambda_{(2,q_2,\ldots,q_k)(q_1-1,q_2,\ldots,q_k)}\right)\\ &\quad\times|\phi'_{(p+1)};r(2,q_2,\ldots,q_k)\rangle^\lambda\\ &=\cdots\\ &=(-1)^{q_1}\left[\frac{q_1}{q_1-1}\right]^{1/2}|\phi'_{(p+1)};r(2,q_2,\ldots,q_k)\rangle^\lambda.\end{aligned} \tag{17}$$

In going through the successive steps leading to the last equality of Eq. (17), we have made use of the fact that terms such as $[P^{(q_1-1)}]^\lambda_{(2,q_2,\ldots,q_k)(q_1,\ldots,q_k)}=0$ since the Yamanouchi basis is sequence adapted to $S_{q_1}\supset S_{q_1-1}\supset\cdots\supset S_1$.[3,15]

Consider, now, the effect of $E_{p+q_1-1\ p+q_1}$ on $|\phi_{(p)};r(q_1-1,q_2,\ldots,q_k)\rangle^\lambda$. We have

$$\begin{aligned}E_{p+q_1-1\ p+q_1}&|\phi_{(p)};r(q_1-1,q_2,\ldots,q_k)\rangle^\lambda\\ &=\sqrt{2}\,[P^{(q_1)}]^\lambda_{(2,q_2,\ldots,q_k)(q_1,q_2,\ldots,q_k)}\times|\phi'_{(p+1)};r(2,q_2,\ldots,q_k)\rangle^\lambda\\ &=E_{p+q_1-1\ p+q_1}\left\{\frac{-1}{q_1-1}\omega^\lambda_{r(q_1-1,q_2,\ldots,q_k)}+\frac{\sqrt{(q_1-2)q_1}}{q_1-1}\omega^\lambda_{r(q_1,q_2,\ldots,q_k)}\right\}\\ &\quad\times|\phi_0\phi^1_{p+1}\cdots\phi^{q_1-1}_{p+q_1}\phi^{q_1}_{p+q_1-1}\cdots\phi^{N-2p}_{N-p}\rangle\\ &=\left\{-\frac{\sqrt{2}}{q_1-1}[P^{(q_1)}]^\lambda_{(2q_2\cdots q_k)(q_1-1\,q_2\cdots q_k)}+(-1)^{q_1}\left[\frac{q_1-2}{q_1-1}\right]^{1/2}\frac{q_1}{q_1-1}\right\}\\ &\quad\times|\phi'_{(p+1)};r(2q_2,\ldots,q_k)\rangle^\lambda,\end{aligned} \tag{18}$$

where the behavior of the Yamanouchi basis under elementary tranpositions and the results of Eq. (17) have been used and $|\phi'_{(p+1)}\rangle$ is as given in Eq. (16). Comparison of the first and last equalities of Eq. (18) and simplification leads to

$$E_{p+q_1-1\ p+q_1}|\phi_{(p)};\ r(q_1-1,q_2,\ldots,q_k)\rangle^\lambda = (-1)^{q_1}\left[\frac{q_1-2}{q_1-1}\right]^{1/2} \times|\phi'_{(p+1)};\ r(2,q_2,\ldots,q_k)\rangle^\lambda. \tag{19}$$

Combining the results of Eqs. (18) and (19), we can also establish that

$$E_{p+q_1\ p+q_1-1}|\phi'_{(p+1)};\ r(2,q_2,\ldots,q_k)\rangle^\lambda = (-1)^{q_1}\left\{\left[\frac{q_1}{q_1-1}\right]^{1/2}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_k)\rangle^\lambda + \left[\frac{q_1-2}{q_1-1}\right]^{1/2}|\phi_{(p)};\ r(q_1-1,q_2,\ldots,q_k)\rangle^\lambda\right\} \tag{20}$$

and

$$E_{p+q_1\ p+q_1+1}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_k)\rangle^\lambda = (-1)^{q_1+1}\left[\frac{q_1-1}{q_1}\right]^{1/2} \times|\phi''_{(p+1)};\ r(2,q_2,\ldots,q_k)\rangle^\lambda, \tag{21}$$

where

$$|\phi''_{(p+1)}\rangle = |\phi_0\phi^1_{p+q_1}\phi^2_{p+q_1}\phi^3_{p+1}\cdots\phi^{N-2p}_{N-p}\rangle \tag{22}$$

if $q_2\neq q_1+1$.

Thus if the contracted orbitals have one of their particle coordinates in the first row of $(q_1,q_2,\ldots q_k)$, then the projected basis yields results similar to those of Harter and Patterson[4] for the canonical Weyl basis. The results obtained here for the matrix elements were given by them[4] just as a set of "rules."

We now assume that for the contraction of orbital pairs $(\phi_{p+q_j-1},\ \phi_{p+q_j})$ $(j=1,2,\ldots,i)$ occurring in $|\phi_{(p)};\ r(q_1,q_2,\ldots,q_k)\rangle^\lambda$ results, similar to those in of Eqs. (17) and (19), are valid. The generalized form of these results for the pair is

$$E_{p+q_j-1\ p+q_j}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_k)\rangle^\lambda = (-1)^{q_j}\left[\frac{q_j-2j+2}{q_j-2j+1}\right]^{1/2} \times|\phi'_{(p+1)};\ r(2,q_1+2,\ldots,q_{j-1}+2,q_{j+1},\ldots,q_k)\rangle^\lambda, \tag{23}$$

$$E_{p+q_j-1\ p+q_j}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_j-1,\ldots,q_k)\rangle^\lambda = (-1)^{q_j}\left[\frac{q_j-2j}{q_j-2j+1}\right]^{1/2} \times|\phi'_{(p+1)};\ r(2,q_1+2,\ldots,q_{j-1}+2,q_{j+1},\ldots,q_k)\rangle^\lambda, \tag{24}$$

where, in view of the orbital numbering used, j is the row index of the subtableau and

$$|\phi'_{(p+1)}\rangle = |\phi_0\phi^1_{p+q_j-1}\phi^2_{p+q_j-1}\phi^3_{p+1}\cdots\phi^{N-2p}_{N-p}\rangle. \tag{25}$$

Using Eqs. (24) and (25) we can also obtain the further results,

$$E_{p+q_j\ p+q_j-1}|\phi'_{(p+1)};\ r(2,q_1+2,\ldots,q_{j-1}+2,q_{j+1},\ldots,q_k)\rangle^\lambda = (-1)^{q_j}\left\{\left[\frac{q_j-2j+2}{q_j-2j+1}\right]^{1/2}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_j,\ldots,q_k)\rangle^\lambda + \left[\frac{q_j-2j}{q_j-2j+1}\right]^{1/2}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_j-1,\ldots,q_k)\rangle^\lambda\right\} \tag{26}$$

and

$$|\phi_{(p)};\ r(q_1,q_2,\ldots,q_j,\ldots,q_k)\rangle^\lambda = \hat\theta_{(p+q_j+1,\,p+q_j)} \times|\phi_{(p)};\ r(q_1,q_2,\ldots,q_j+1,q_{j+1},\ldots,q_k)\rangle^\lambda, \tag{27}$$

where

$$\hat\theta_{(p+q_j+1,\ p+q_j)} = [(q_j-2j+3)(q_j-2j+1)]^{-1/2} \times\{(q_j-2j+2)E_{p+q_j+1\ p+q_j}E_{p+q_j\ p+q_j+1}-(q_j-2j+3)\}, \tag{28}$$

for $q_{j-1}<q_j\leq q_{j+1}-1$.

Using Eqs. (23), (24), and (26)–(28) we now attempt to obtain the results of the contraction of the orbital pair $(\phi_{p+q_{i+1}-1},\ \phi_{p+q_{i+1}})$ occurring in $|\phi_{(p)};\ r(q_1,\ldots,q_k)\rangle^\lambda$. Consider $E_{p+q_{i+1}-1\ p+q_{i+1}}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_i,q_{i+1},\ldots,q_k)\rangle^\lambda$. Let $q_i=q_{i+1}-m$. If $m=1$, it can be readily shown that this expression vanishes. If $m\neq1$, we have, using Eqs. (27) and (28), the result,

$$E_{p+q_{i+1}-1\ p+q_{i+1}} \times|\phi_{(p)};\ r(q_1,q_2,\ldots,q_i,q_{i+1},\ldots,q_k)\rangle^\lambda = E_{p+q_{i+1}-1\ p+q_{i+1}}\hat\theta_{(p+q_{i+1}-m+1,\,p+q_{i+1}-m} \times|\phi_{(p)};\ r(q_1,\ldots,q_{i+1}-m+1,q_{i+1},\ldots,q_k)\rangle^\lambda. \tag{29}$$

If $m=2$, we terminate the above sequence at Eq. (29). If now we proceed further until we finally obtain the state $|\phi_{(p)};\ r(q_1,q_2,\ldots,q_{i+1}-1,q_{i+1},\ldots,q_k)\rangle$ as follows:

$$E_{p+q_{i+1}-1\ p+q_{i+1}} \times|\phi_{(p)};q_1,q_2,\ldots,q_{i+1}-m,q_{i+1},\ldots,q_k)\rangle^\lambda = E_{p+q_{i+1}-1\ p+q_{i+1}}\prod_{\substack{l=m\\ \text{ordered}}}^{2}\hat\theta_{(p+q_{i+1}-l+1,\,p+q_{i+1}-l)} \times|\phi_{(p)};\ r(q_1,\ldots,q_{i+1}-1,q_{i+1},\ldots,q_k)\rangle^\lambda, \tag{30}$$

where the ordered product implies that the operator corresponding to $l=j$ occurs at the extreme left followed by the next lower value of l and so on. For $m>2$, $E_{p+q_{i+1}\ p+q_{i+1}}$ commutes with each of the operators in the product on the right of Eq. (30) except $\hat\theta_{(p+q_{i+1}-1\ p+q_{i+1})}$. Further $E_{p+q_{i+1}-1\ p+q_{i+1}}|\phi_{(p)};\ r(q_1,q_2,\ldots,q_{i+1}-1,q_{i+1},\ldots,q_k)\rangle^\lambda=0$. Using these results and

Eq. (28) we have

$$E_{p+q_{i+1}-1\; p+q_{i+1}} \times |\phi_{(p)};\, r(q_1, q_2, \ldots, q_{i+1}-m, q_{i+1}, \ldots, q_k)\rangle^{\lambda}$$

$$= (-1)^{q_{i+1}-1}\left[\frac{q_{i+1}-2i}{q_{i+1}-2i-1}\right]^{1/2} \times \prod_{\substack{l=m \\ \text{(ordered)}}}^{3} \hat{\theta}_{(p+q_{i+1}-l+1,\, p+q_{i+1}-l)} E_{p+q_{i+1}-1\; p+q_{i+1}-2}$$

$$\times E_{p+q_{i+1}-1\; p+q_{i+1}} |\phi''_{(p+1)};\, r(2, q_1+2, \ldots, q_{i-1} + 2, q_{i+1}, \ldots, q_k)\rangle^{\lambda}, \qquad (31)$$

where

$$|\phi''_{(p+1)}\rangle = |\phi_0 \phi^1_{p+q_{i+1}-2} \phi^2_{p+q_{i+1}-2} \times \phi^3_{p+1} \cdots \phi^{N-2p}_{N-p}\rangle. \qquad (32)$$

The orbital $\phi_{p+q_{i+1}-1}$ is unoccupied in the product given in Eq. (32). Hence, using Eq. (10) we find that $E_{p+q_{i+1}-1\; p+q_{i+1}}$ just replaces $\phi_{p+q_{i+1}}$ by $\phi_{p+q_{i+1}-1}$. In the resulting state $\phi_{p+q_{i+1}-2}$ is doubly occupied and $\phi_{p+q_{i+1}-1}$ is singly occupied. The use of Eq. (8), therefore, implies that $E_{p+q_{i+1}-1\; p+q_{i+1}-2}$ just leads to the reversal of the respective occupancies followed by a multiplication of the resulting state by (-1). Thus,

$$E_{p+q_{i+1}-1\; p+q_{i+1}} |\phi_{(p)};\, r(q_1, q_2, \ldots, q_k)\rangle^{\lambda}$$

$$= (-1)^{q_{i+1}}\left[\frac{q_{i+1}-2(i+1)+2}{q_{i+1}-2(i+1)+1}\right]^{1/2} \times \prod_{\substack{l=m \\ \text{(ordered)}}}^{3} \hat{\theta}_{(p+q_{i+1}-l+1,\, p+q_{i+1}-l)}$$

$$\times |\phi'''_{(p+1)};\, r(2, q_1+2, q_2+2, \ldots, q_{i-1}+2, q_{i+1}, \ldots, q_k)\rangle^{\lambda}, \qquad (33)$$

where

$$|\phi'''_{(p+1)}\rangle = |\phi_0 \phi^1_{p+q_{i+1}-1} \phi^2_{p+q_{i+1}-1} \cdots \phi^{N-2p}_{N-p}\rangle. \qquad (34)$$

Let us now consider the effect of $\hat{\theta}_{(p+q_{i+1}-2,\, p+q_{i+1}-3)}$ on the state given on the right of Eq. (32). The generator $E_{p+q_{i+1}-3\; p+q_{i+1}-2}$ contained in this operator can yield a zero for this state only if $q_{i-1}+2 = q_{i+1}-1$. This, in turn, implies that $q_i = q_{i+1}-1$ or $q_{i+1}-2$. The first possibility yields zero and has already been omitted. If the second possibility is true, the right of Eq. (30) consists of just one term whose effect is given by Eq. (33) omitting the product of operators.

Thus

$$\hat{\theta}_{(p+q_{i+1}-2,\, p+q_{i+1}-3)} |\phi'''_{(p+1)};\, r(q_1, \ldots, q_k)\rangle^{\lambda} \neq 0.$$

In these cases the effect of the generators contained in the operators on the singly occupied portion of $|\phi'''_{(p+1)}\rangle$ is assumed known. Hence after simplification we have the result

$$\hat{\theta}_{(p+q_{i+1}-2,\, p+q_{i+1}-3)} \times |\phi'''_{(p+1)};\, r(2, q_1+2, \ldots, q_{i-1}+2, q_{i+1}, \ldots, q_k)\rangle^{\lambda}$$

$$= |\phi'''_{(p+1)};\, r(2, q_1+2, \ldots, q_{i-1}+2, q_{i+1}-1, \ldots, q_k)\rangle^{\lambda}. \qquad (35)$$

Using similar arguments for the other operators in the product on the right of Eq. (33), we finally obtain

$$E_{p+q_{i+1}-1\; p+q_{i+1}} |\phi_{(p)};\, r(q_1, q_2, \ldots, q_i, q_{i+1}, \ldots, q_k)\rangle^{\lambda}$$

$$= (-1)^{q_{i+1}}\left[\frac{q_{i+1}-2(i+1)+2}{q_{i+1}-2(i+1)-1}\right]^{1/2} |\phi'''_{(p+1)};\, r(2, q_1+2, \ldots, q_{i-1}+2, q_i+2, \ldots, q_k)\rangle^{\lambda}, \qquad (36)$$

where we have replaced $q_{i+1} - m + 2$ by $q_i + 2$.

Thus we find that the contraction of the orbitals $(\phi_{p+q_{i+1}-1}, \phi_{p+q_{i+1}})$ yields a result similar to that given by Eq. (23) weighted by a coefficient which depends on the "city block distance.[4]" The other results for this orbital pair can be obtained in the same way.

3. DISCUSSION

The transformation properties established in Eqs. (8) and (10) were relatively easy to obtain because the matching permutation in these cases was the identity in the corresponding Eqs. (5) and (4), respectively. Hence the standard tableau index was not changed. This is not generally true for the results in Eqs. (23), (24), (26), and (27). In order to take account of this, the transformation of Eq. (27) had to be considered first before obtaining the inductive proof leading to Eq. (33).

The results of Sec. 2 were obtained for the restricted class of tensor products of the type given in Eq. (7) rather than the more general ones of Eq. (1). This, however, is not a serious limitation. The ordering of the orbitals given in Eq. (1) ensures that the transformation given in Eq. (8) requires only the identity permutation. The proof for this case, then goes through as before. Similar considerations apply to the case given in Eq. (10). The transformations given in Eqs. (23) and (24) involve only singly occupied orbitals. Hence we could have replaced all the orbitals in this portion of Eq. (1) by a natural sequence which includes the reference orbitals and retrieved the rest after the transformation has been effected.

The projected basis, $|\phi_{(p)};\, rs\rangle^{\lambda}$, of Eq. (7) permits a ready identification with the Weyl tableau basis.[4] For such a basis function [cf. Eq. (7)] having complete ordering of the basis orbitals in the tensor product, the corresponding Gel'fand tableau is also uniquely defined. As an illustration, consider $|\phi^1_1\phi^2_1\phi^3_2\phi^4_3\phi^5_5;\, r(2,5)\rangle^{[2^2,1]}$ of U(5). This basis function corresponds to the Weyl tableau

1	1
2	5
3	

and is, in turn, uniquely identifiable with the Gel'fand tableau[13]

$$\left|\begin{array}{ccccccccc} 2 & & 2 & & 1 & & 0 & & 0 \\ & 2 & & 1 & & 1 & & 0 & \\ & & 2 & & 1 & & 1 & & \\ & & & 2 & & 1 & & & \\ & & & & 2 & & & & \end{array}\right\rangle .$$

On the other hand, suppose we have a projected basis of the type given in Eq. (1). We can then relate it simply to the nearest completely ordered basis through elementary generators and identify the latter uniquely as above. Consider $|\phi_2^1\phi_2^2\phi_1^3\phi_3^4\phi_4^5;\, r(2,5)\rangle^{[2^2,1]}$ of U(5). We start with $|\phi_1^1\phi_1^2\phi_2^3\phi_3^4\phi_4^5;\, r(2,5)\rangle^{[2^2,1]}$ of U(5) which has the correspondence with the Weyl and Gel'fand basis

$$\begin{array}{cc} 1 & 1 \\ 2 & 4 \\ 3 & \end{array} \Rightarrow \left|\begin{array}{ccccccccc} 2 & & 2 & & 1 & & 0 & & 0 \\ & 2 & & 2 & & 1 & & 0 & \\ & & 2 & & 1 & & 1 & & \\ & & & 2 & & 1 & & & \\ & & & & 2 & & & & \end{array}\right\rangle ,$$

respectively.

Using Eq. (8) we have then

$$|\phi_2^1\phi_2^2\phi_1^3\phi_3^4\phi_4^5;\, r(2,5)\rangle^{[2^2,1]} = -E_{21}\,|\phi_1^1\phi_1^2\phi_2^3\phi_3^4\phi_4^5;\, r(2,5)\rangle^{[2^2,1]}.$$

In view of this result and the matrix elements of the Gel'fand basis, we can obtain the unique identification

$$|\phi_2^1\phi_2^2\phi_1^3\phi_3^4\phi_4^5;\, r(2,5)\rangle^{[2^2,1]} = -\left|\begin{array}{ccccccccc} 2 & & 2 & & 1 & & 0 & & 0 \\ & 2 & & 2 & & 1 & & 0 & \\ & & 2 & & 1 & & 1 & & \\ & & & 2 & & 1 & & & \\ & & & & 1 & & & & \end{array}\right\rangle .$$

The existence of such equivalences imply that the programmed projected basis set[6,7] provides the most direct method for determining the matrix elements of the generators of U(n) over the standard bases[1,13] spanning the IR $\{2^{N/2-S}, 1^{2S}\}$.

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The Lorentz group in the oscillator realization. I. The group SO(2,1) and the transformation matrices connecting the SO(2) and SO(1,1) bases

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The unitary transformation connecting the SO(2) and SO(1,1) bases for the principal and discrete series of representations of the three-dimensional Lorentz group is determined by using the oscillator representation technique. The Hilbert space and the SO(1,1) basis, in this realization, have a simple appearance while the compact basis appears as the solution of an ordinary differential equation reducible to the confluent hypergeometric equation by a simple substitution. The Taylor expansion of this solution obtained by the use of certain functional identities yields the continuous spectrum of the SO(1,1) representations and the unitary transformation from the compact to the noncompact basis after the Sommerfeld–Watson transformation.

1. INTRODUCTION

The unitary irreducible representations (UIR's) of the group SO(2, 1) can be studied either through the finite group[1] or through the self-adjoint representation of the Lie algebra.[2] The representations in the former method deal with the finite group elements and there is no occasion to consider a specific subgroup in the reduced form. A more detailed description which is of interest for further applications of the theory, however, requires an explicit construction of the generators of the Lie algebra in suitable Hilbert space and of the matrix elements of operators corresponding to the group elements. An investigation aimed at analyzing such a level of detail proceeds by reducing the representations under the compact[3] or the noncompact[4] subgroup.

A class of realizations of this genre is the so-called oscillator representation, the discovery of which dates back to the pioneering work of Schwinger[5] in the early fifties. In this method the generators of the group in a given UIR are constructed out of a set of harmonic oscillator creation and annihilation operators in the coordinate representation. The Hilbert space, in this realization, has an especially simple form and consists of square integrable functions defined on the plane.

Holman and Biedenharn[6] derived such operators for the discrete series of UIR's of SO(2, 1) and used them to solve the Clebsch–Gordan problem for the group. Mukunda and Radhakrishnan[7] generalized these operators to include the principal series of UIR's and used them to determine the Clebsch–Gordan coefficients of SO(2, 1) in a continuous basis. Recently Wolf,[8] Boyer,[9] Moshinsky[10] and co-workers have established very similar realizations of the generators of the Lie algebra from entirely different considerations and have obtained some important results on their exponentiation to the group and on the representation spaces in which they act.

It is, therefore, interesting to examine some aspects of the UIR's of SO(2, 1) in the light of these developments. The object of this paper is to show that the oscillator realization can be used in a simple and unitary way to evaluate the transformation coefficients from the compact SO(2) to the noncompact SO(1, 1) basis for any UIR of SO(2, 1) belonging to the principal or the discrete series. Our method is similar to that of a previous paper[11] in which we studied the reduction of the UIR's of SO(3, 1) in the SO(2, 1) basis. In this method, a certain degree of uniformity is achieved in the treatment of the discrete class of UIR's on the one hand, and the continuous class of UIR's, on the other. Similar problems have been considered by Vilenkin[12] (V), and recently by Kalnins[13](K), and by Montgomery and O' Raifeartaigh[14] (MO'R) amongst others.[15] While V obtains the finite transformation matrices, K determines the mixed basis and the overlap matrix elements for the principal series alone. MO'R, on the other hand, obtains the overlap functions between SO(2) and a nonsubgroup basis. Our result for the principal series [Eq. (2.24)] agrees with that of K. We, however, go, to some extent, beyond the work of our predecessor. While the principal series alone is considered by K, we have succeeded in finding the transformation coefficients for both the principal and the discrete series [Eq. (3.9)].

2. THE PRINCIPAL SERIES

We start with a few remarks concerning the group SO(2, 1) or its covering group SU(1, 1). The group has three infinitesimal operators, J_0 generating plane rotation, and J_1, J_2 generating pure Lorentz transformations. In a unitary representation these are Hermitian and satisfy the commutation relation (CR),

$$[J_\mu, J_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda, \qquad \mu,\nu,\lambda=(0,1,2), \tag{2.1}$$

A possible solution to CR Eq. (2.1) is obtained by the choice[7]

$$\begin{aligned} J_0 &= \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2),\\ J_1 &= \tfrac{1}{4}[(a_1{}^\dagger)^2 + (a_1)^2 + (a_2^\dagger)^2 + (a_2)^2],\\ J_2 &= \frac{i}{4}\,[(a_1^\dagger)^2 - (a_1)^2 + (a_2^\dagger)^2 - (a_2)^2], \end{aligned} \tag{2.2}$$

where $a_i^\dagger$ and a_i are the familiar harmonic oscillator creation and annihilation operators in the coordinate representation,

$$a_i = -\frac{i}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right), \quad a_i^\dagger = \frac{i}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right). \tag{2.3}$$

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The operators (2.2) are Hermitian under the scalar product,

$$(f,g)=\int\int dx_1dx_2 f^*(x_1,x_2)g(x_1,x_2). \tag{2.4}$$

The Casimir operator has the form

$$Q=J_0^2-J_1^2-J_2^2=-(\tfrac{1}{4}+S^2),$$

where S is a Hermitian operator given by

$$S=-\frac{i}{2}(a_1^\dagger a_2^\dagger - a_1a_2). \tag{2.5}$$

The construction (2.2), therefore, leads to the principal series of UIR of SO(2, 1).

To obtain the explicit form of SO(2) or SO(1, 1) bases we need to consider the simultaneous eigenstates of one of the generators J_0 or J_2 and the Casimir invariant S. This is achieved by constructing the appropriate eigenvalue equations in the regions D_1 and D_2 corresponding to $x_2\geq 0,\ -x_2\leq x_1\leq x_2$ and $x_1\geq 0,\ -x_1\leq x_2\leq x_1$, respectively. In D_1 we introduce $x_1=r\sinh\eta,\ x_2=r\cosh\eta$; the corresponding transformation in D_2 is obtained by interchanging x_1 and x_2. Both in D_1 and D_2 the operator S has the form

$$S=-\frac{i}{2}\frac{\partial}{\partial\eta}. \tag{2.6}$$

In a given UIR of the principal series, we, therefore, need to consider functions of the form

$$f^{D_j}(x_1,x_2)=\frac{i}{\sqrt{2\pi}}e^{2is\eta}f_s^{D_j}(r). \tag{2.7}$$

The generators can then be rewritten as one-variable operators in D_j:

$$J_0^{(D_j)}=\frac{\epsilon_j}{4}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{4s^2}{r^2}-r^2\right),$$
$$J_2^{(D_j)}=-\frac{i}{2}\left(r\frac{d}{dr}+1\right), \tag{2.8}$$

where $\epsilon_1=-\epsilon_2=1$. The eigenfunction of the compact generator in D_j belonging to the eigenvalue m is obtained by setting

$$x=r^2,\quad f_{ms}^{D_j}=x^{is}e^{-x/2}y_{ms}^{D_j}(x).$$

The function $y_{ms}^{D_j}$ is then a solution of the confluent hypergeometric equation which has two linearly independent solutions,

$$y_{ms}^{D_j}(x)={}_1F_1(\epsilon_j m+\tfrac{1}{2}+is;\ 1+2is;x),$$
$$z_{ms}^{D_j}(x)=x^{-2is}\ {}_1F_1(\epsilon_j m+\tfrac{1}{2}-is;\ 1-2is;x),$$
$$m=0,\pm\tfrac{1}{2},\pm 1,\ldots. \tag{2.9}$$

It is evident that the solutions should be chosen such that they fulfill the requirement of orthonormality. A simple test shows that none of the solutions (2.9) constitute an orthogonal set and these are, therefore, unsuitable for use as basis functions. However, as we shall see presently, an orthogonal set can easily be constructed by taking insead of one solution a linear combination of the first and second solution of the confluent hypergeometric equation which are given by Eq. (2.9). The appropriate linear combinations in D_1 and D_2 turn out to be

$$f_{ms}^{D_1}\equiv u_{ms}=e^{-x/2}\left[\frac{\Gamma(-2is)\Gamma(m+\frac{1}{2}+is)}{\Gamma(m+\frac{1}{2}-is)}x^{is}\ {}_1F_1(m+\tfrac{1}{2}+is;\ 1+2is;x)+\Gamma(2is)x^{-is}\ {}_1F_1(m+\tfrac{1}{2}-is;\ 1-2is;x)\right], \tag{2.10}$$

$$f_{ms}^{D_2}=u_{-ms}. \tag{2.11}$$

To establish the orthogonality of the eigenfunctions we first rewrite the scalar product (2.4) with respect to which the solutions are required to be orthogonal in a slightly different form. Using Eq. (2.7) and noting that the Hilbert space consists of functions of either even or odd parity according as we restrict ourselves to a continuous integral or half-integral class of representations, we obtain

$$(f_{ms},f_{nt})=\delta(s-t)\langle m|n\rangle, \tag{2.12}$$

where $\langle m|n\rangle$ stands for the following expression,

$$\langle m|n\rangle=\int_0^\infty u_{ms}^* u_{ns} r\,dr+\int_0^\infty u_{-ms}^* u_{-ns} r\,dr. \tag{2.13}$$

The rhs of the above expression can be evaluated in the traditional way by using the Sturm—Liouville theory of the second order differential equations and we have,

$$\langle m|n\rangle=\frac{1}{4(m-n)}\lim_{\substack{b\to\infty\\ a\to 0}}\left|\left(u_{ns}r\frac{du_{ms}^*}{dr}-u_{ms}^* r\frac{du_{ns}}{dr}\right)-\left(u_{-ns}r\frac{du_{-ms}^*}{dr}-u_{-ms}^* r\frac{du_{-ns}}{dr}\right)\right|_a^b. \tag{2.14}$$

Using the asymptotic form of the confluent hypergeometric function and its behavior near the origin we finally obtain

$$\langle m|n\rangle=\delta_{mn}\frac{\pi^2}{2g(s)}, \tag{2.15}$$

where $g(s)$ is equal to $\cosh^2\pi s$ or $\sinh^2\pi s$ according as we consider the principal series of the UIR belonging to the integral or half-integral class.

Since J_2 is a first order differential operator and retains the same form in D_1 and D_2, the normalized SO(1, 1) bases are given by

$$f_{\mu s}^{D_j}=\frac{1}{\sqrt{\pi}}x^{i\mu-1/2},\quad -\infty<\mu<\infty. \tag{2.16}$$

It must be emphasized that unlike the SO(1, 1) bases $f_{\mu s}^{D_j}$ which are orthogonal separately in D_j, the SO(2) bases are orthogonal in the entire x_1-x_2 plane spanned by D_1 and D_2. The former is a consequence of the double multiplicity of SO(1, 1) representations contained in SO(2, 1).

To determine the unitary transformation connecting the two bases we expand the functions $u_{\pm ms}$ of the SO(2) basis by using the Burchnall and Chaundy formula,[16]

$$e^{-x/2}\ {}_1F_1(a;c;x)=\sum_{n=0}^{\infty}\frac{(-)^n}{n!\,2^n}\ {}_2F_1\left[\begin{matrix}a,\ -n\\ c\ \ ;2\end{matrix}\right]x^n. \tag{2.17}$$

After simplification, the process finally yields

$$u_{\pm ms}=\Gamma(\pm m+\tfrac{1}{2}+is)(X_{\pm m}^s+X_{\pm m}^{-s}), \tag{2.18}$$

where

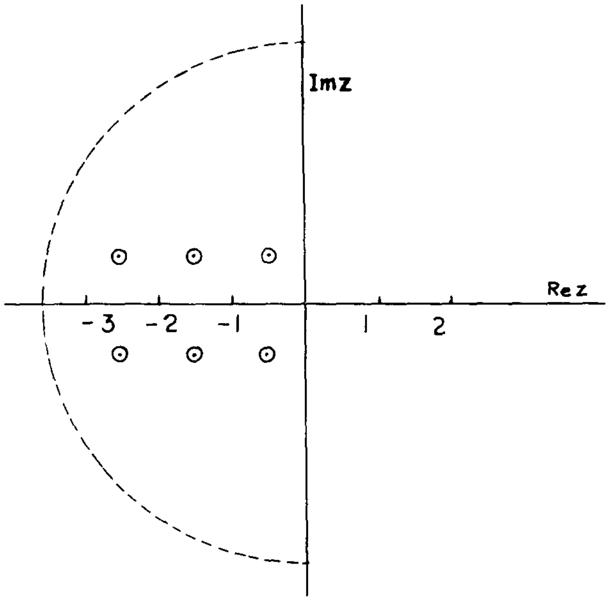

FIG. 1.

$$X^s_{\pm m} = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\Gamma(-2is-n)}{\Gamma(\pm m+\frac{1}{2}-is-n)} 2^{-(\pm m+1/2+is+n)}$$
$$\times\, {}_2F_1\left[\begin{matrix}\pm m+\frac{1}{2}-is, & \pm m+\frac{1}{2}+is\\ \pm m+\frac{1}{2}-is-n & \end{matrix};\tfrac{1}{2}\right] x^{n+is}. \tag{2.19}$$

The expansion (2.18) of the SO(2) basis function has desired analytic form but contains inadmissible values of μ. To circumvent this difficulty we express the sum as a contour integral in the complex z plane ($z=i\mu$) and apply the Sommerfeld–Watson transformation. The various terms in the sum are easily recognized as the residues at $z=\mp is-\frac{1}{2}-n$ ($n=0,1,2,\ldots$) of the analytic function

$$\chi_{\pm m}(z) = \frac{\Gamma(z+is+\frac{1}{2})\,\Gamma(z-is+\frac{1}{2})}{\Gamma(\pm m+z+1)} 2^{\mp m+z}$$
$$\times\, {}_2F_1\left[\begin{matrix}\pm m+\frac{1}{2}-is, & \pm m+\frac{1}{2}+is\\ \pm m+z+1 & \end{matrix};\tfrac{1}{2}\right] x^{-z-1/2}. \tag{2.20}$$

Since for fixed $|z|<1$, $[1/\Gamma(c)]\,{}_2F_1(a,b;c;z)$ is an entire function of the parameters, $\chi_{\pm m}(z)$ is a meromorphic function going to zero rapidly as $|z|$ tends to infinity in the region $\mathrm{Re}z<0$. The singularities of $\chi_{\pm m}(z)$ are simple poles arising from the Γ functions in the factor

$$\Gamma(z+is+\tfrac{1}{2})\,\Gamma(z-is+\tfrac{1}{2})$$

and, as shown in Fig. 1, are located at the points

$$z=\mp is-\tfrac{1}{2}-n,\quad n=0,1,2,\ldots .$$

Let us now choose a contour C consisting of the infinite semicircle S on the left and the pure imaginary axis. The singularities enclosed by the contour are the simple poles mentioned above. Therefore, by Cauchy's theorem,

$$\frac{1}{2\pi i}\oint dz\,\chi_{\pm m}(z) = \sum_{n=0}^{\infty}\mathrm{Res}\Big[\chi_{\pm m}(z)\Big]_{z=-is-1/2-n} + \sum_{n=0}^{\infty}\mathrm{Res}\Big[\chi_{\pm m}(z)\Big]_{z=+is-1/2-n}. \tag{2.21}$$

The first and the second terms on the rhs by our previous analysis, are respectively equal to $X^s_{\pm m}$ and $X^{-s}_{\pm m}$ and the integral on the lhs, as can be easily verified, vanishes on S. The equation can, therefore, be written as

$$u_{\pm ms} = \frac{\Gamma(\pm m+\frac{1}{2}+is)}{2\pi i}\int_{-i\infty}^{i\infty}\chi_{\pm m}(z)\,dz. \tag{2.22}$$

Setting $z=-i\mu$ and rearranging the terms, the above equation can finally be written in the form

$$u_{\pm ms} = \frac{\Gamma(\pm m+\frac{1}{2}+is)}{2\sqrt{\pi}}\int_{-\infty}^{\infty} d\mu\, \frac{\Gamma(-i\mu+is+\frac{1}{2})\Gamma(-i\mu-is+\frac{1}{2})}{\Gamma(\pm m-i\mu+1)}$$
$$\times\, 2^{-i\mu\mp m}\,{}_2F_1\left[\begin{matrix}\pm m+\frac{1}{2}+is, & \pm m+\frac{1}{2}-is\\ \pm m-i\mu+1 & \end{matrix};\tfrac{1}{2}\right] f_{\mu s}(x). \tag{2.23}$$

This formula immediately leads to the following expression for the transformation coefficient for the UIR's of the continuous integral class,

$$(f_{ms}, f_{\mu t}) = \delta(s-t)\frac{\cosh\pi s}{\pi\sqrt{2\pi}}$$
$$\times\Bigg(\frac{\Gamma(m+\frac{1}{2}+is)\Gamma(-i\mu+is+\frac{1}{2})\Gamma(-i\mu-is+\frac{1}{2})}{\Gamma(m-i\mu+1)}$$
$$\times\, 2^{-i\mu-m}\,{}_2F_1\left[\begin{matrix}m+\frac{1}{2}+is, & m+\frac{1}{2}-is\\ m-i\mu+1 & \end{matrix};\tfrac{1}{2}\right] + (m \longleftrightarrow -m)\Bigg). \tag{2.24}$$

For representations belonging to the continuous half-integral class the same expression holds with $\cosh\pi s$ replaced by $\sinh\pi s$. Equation (2.24) exactly agrees with the overlap functions for the principal series obtained by Kalnins[13] from entirely different considerations.

3. THE DISCRETE SERIES $D_k^{\pm}$

To obtain the representations belonging to the discrete class $D_k^{\pm}$, we consider in place of Eqs. (2.2) the following set of generators,[6-8]

$$J_0 = \tfrac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1),$$
$$J_1 = \tfrac{1}{4}[(a_1^\dagger)^2 + (a_1)^2 + (a_2^\dagger)^2 + (a_2)^2], \tag{3.1}$$
$$J_2 = -\frac{i}{4}[(a_1^\dagger)^2 - (a_1)^2 + (a_2^\dagger)^2 - (a_2)^2].$$

These operators are again Hermitian under the scalar product (2.4) and since the eigenvalues of J_0 are positive definite the construction will lead to the positive discrete series D_k^+. The corresponding generators for D_k^- are obviously obtained by replacing J_0 and J_1 by $-J_0$ and $-J_1$, respectively. As there is no essential difference between D_k^+ and D_k^- we shall exhibit the details for D_k^+.

The Hilbert space for D_k^+ consists of all functions of the form

$$f(x_1,x_2) = \frac{1}{\sqrt{2\pi}}\,e^{i(2k-1)\theta} f(r), \tag{3.2}$$

where r, θ are the usual polar coordinates. The operators (3.1) can then be expressed in the form

$$J_0=\tfrac{1}{4}\left[r^2-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{(2k-1)^2}{r^2}\right],$$
$$J_2=-\frac{i}{2}\left[r\frac{d}{dr}+1\right]. \tag{3.3}$$

The normalized SO(2) bases are now given by

$$\Psi_m=\frac{\sqrt{2}}{\Gamma(2k)}\left(\frac{\Gamma(m+k)}{\Gamma(m-k+1)}\right)^{1/2}e^{-x/2}$$
$$\times\ x^{k-1/2}\ {}_1F_1(-m+k;\ 2k;\ x), \tag{3.4}$$

where $x=r^2$, $m=k,\ k+1,\ \ldots$.

The normalized SO(1,1) bases, on the other hand, are

$$f_{\mu k}(x)=\frac{1}{\sqrt{\pi}}\ x^{i\mu-1/2}. \tag{3.5}$$

Using the technique of the previous section we now have for D_k^+,

$$\Psi_{mk}=\frac{\sqrt{2}}{\sqrt{\Gamma(m-k+1)\Gamma(m+k)}}\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\frac{\Gamma(m+n+k)}{\Gamma(2k+n)}$$
$$\times{}_2F_1\begin{bmatrix}-m-k+1, & -m+k & \\ & & ;\tfrac{1}{2}\\ -n-m-k+1 & & \end{bmatrix}(2)^{m-k-n}x^{n+k-1/2}. \tag{3.6}$$

The series on the rhs can now be regarded as the sum of the residues at $z=-k-n$ $(n=0,1,2,\ldots)$ of the analytic function

$$\chi(z)=\frac{\Gamma(z+k)\Gamma(m-z)2^{m+z}}{\Gamma(k-z)}$$
$$\times{}_2F_1\begin{bmatrix}-m+k, & -m-k+1 & \\ & & ;\tfrac{1}{2}\\ z-m+1 & & \end{bmatrix}x^{-z-1/2}. \tag{3.7}$$

Note that the parameters of the hypergeometric function appearing on the rhs are such $(|a|<|c|)$ that this is nonsingular even when z is a negative integer. The only singularities of $\chi(z)$ that are enclosed by the contour C of the previous section are, therefore, simple poles at $z=-k-n$ $(n=0,1,2,\ldots)$. Since the integral on the semicircular part of C again vanished we have

$$\Psi_{mk}=\frac{1}{\sqrt{\Gamma(m-k+1)\Gamma(m+k)(2\pi)}}$$
$$\times\int_{-\infty}^{\infty}d\mu\frac{\Gamma(k-i\mu)\Gamma(m+i\mu)2^{m-i\mu}}{\Gamma(k+i\mu)}$$
$$\times{}_2F_1\begin{bmatrix}-m+k, & -m-k+1 & \\ & & ;\tfrac{1}{2}\\ -i\mu-m+1 & & \end{bmatrix}f_{\mu k}(x). \tag{3.8}$$

The transformation coefficients are now given by

$$(\Psi_{mk},f_{\mu k'})=\frac{\delta_{kk'}}{\sqrt{2\pi}}\frac{\Gamma(k-i\mu)\Gamma(m+i\mu)}{\sqrt{\Gamma(m-k+1)\Gamma(m+k)}}$$
$$\times\frac{2^{m-i\mu}}{\Gamma(k+i\mu)}\ {}_2F_1\begin{bmatrix}-m+k, & -m-k+1 & \\ & & ;\tfrac{1}{2}\\ -i\mu-m+1 & & \end{bmatrix}. \tag{3.9}$$

Equation (3.9) strongly resembles the overlap function (2.24) for the principal series.

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A simple derivation of the Onsager–Machlup formula for one-dimensional nonlinear diffusion process

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Transforming the Fokker–Planck equation into a self-adjoint form the Onsager–Machlup formula for a one-dimensional nonlinear diffusion process is derived rigorously. By approximating the Wiener measure by an n-gate cylinder measure, an equation of motion for the most probable path is also derived.

I. INTRODUCTION

Motivated from the monumental works due to Wiener[1] (Wiener process) and Onsager and Machlup[2] (linear diffusion process), a program (i.e., the Onsager–Machlup formula) to represent the transition probability law of a nonlinear diffusion process in terms of integration over sample paths has been pursued by several authors.[3–5] Although they obtained the generalized version of the Onsager–Machlup formula, their derivations can not be justified mathematically because they started with a formal expression of the Wiener measure. Therefore, rigorous derivation of the Onsager–Machlup formula for a general nonlinear diffusion process seems to still be an open problem.

In the present paper we derive the Onsager–Machlup formula for a one-dimensional nonlinear diffusion process from a fundamental point of view:

First of all, we transform the Fokker–Planck equation, which determines the probabilistic behavior of the diffusion process completely, into a self-adjoint form. Then making use of the Feynman–Kac formula we obtain a rigorous expression of the Onsager–Machlup formula.

Secondly, approximating the Wiener measure which appears in the Feynman–Kac formula by an n-gate cylinder measure we obtain a more familiar expression of the Onsager–Machlup formula and derive an equation of motion for the most probable path.

II. ONSAGER-MACHLUP FORMULA

By the notion of a one-dimensional nonlinear diffusion process we denote a stochastic process $X=X(t)$ on the real line $\mathbb{R}$ described by the stochastic differential equation of Itô type,

$$dX(t)=a(X(t))\,dt+b(X(t))\,dB(t), \tag{1}$$

where $B=B(t)$ denotes a standard Brownian motion (i.e., a Wiener process with diffusion constant equal to unity), $a=a(x)$ a drift velocity, and $b=b(x)$ a diffusion coefficient. A special case with $a=0$ and $b=\sqrt{D}$ (D; a positive constant) is known to be a Wiener process with diffusion constant D. $a=-\beta x$ and $b=\sqrt{D}$ (β; a positive constant) lead to the case of a linear diffusion process (i.e., a Smoluchowski process) with damping constant β and diffusion constant D.

The stochastic differential equation (1) is equivalent to the following Fokker–Planck equation,

$$\dot{p}(x,t\,|\,y,u)=-\frac{\partial}{\partial x}[a(x)p(x,t\,|\,y,u)]+\frac{\partial^2}{\partial x^2}[b^2(x)p(x,t\,|\,y,u)], \tag{2}$$

where $p=p(x,t\,|\,y,u)$ with $t>u$ denotes a transition probability density of the process $X=X(t)$.

In the case of a Wiener process making use of the Chapman–Kolmogorov relation and an elementary solution of Eq. (2) with $a=0$ and $b=\sqrt{D}$,

$$p_w(x,t\,|\,y,u)=\frac{1}{\sqrt{4\pi D(t-u)}}\exp\left(-\frac{(x-y)^2}{4D(t-u)}\right), \tag{3}$$

we can represent the transition probability density in terms of integration over continuous sample paths

$$p_w(x,t\,|\,y,u)=\int_{\Omega\binom{x}{t}|\binom{y}{u}}\mu_D(d\gamma), \tag{4}$$

where μ_D denotes a Wiener measure[6] with diffusion constant D and $\Omega\binom{x}{t}|\binom{y}{u}$ a totality of continuous paths γ's starting from y at a time u and arriving at x at t.[1]

In the case of a linear diffusion process Onsager and Machlup[2] derived a similar but rather approximative formula (i.e., the Onsager–Machlup formula) starting with an elementary solution

$$p_s(x,t\,|\,y,u)=\left(\frac{\beta}{2\pi D(1-\exp[-2\beta(t-u)])}\right)^{1/2}\times\exp\left(\frac{\beta}{2D}\frac{\{x-y\exp[-\beta(t-u)]\}^2}{1-\exp[-\beta(t-u)]}\right). \tag{5}$$

In the case of a nonlinear diffusion process $X=X(t)$ we can no longer follow the same procedure as Wiener and Onsager and Machlup because it is difficult to find an elementary solution of the Fokker–Planck equation (2). Therefore, we approach the Onsager–Machlup formula for a one-dimensional nonlinear diffusion process $X=X(t)$ from a fundamental point of view.

If we introduce a stochastic process $Z=Z(t)$ by a stochastic differential equation

$$dZ(t)=b(X(t))^{-1}\,dX(t), \tag{6}$$

we can transform Eq. (1) into

$$dZ(t)=c(Z(t))\,dt+dB(t), \tag{7}$$

where $c(Z(t))=[a(X(t))-\frac{1}{2}b'(X(t))b(X(t))]/b(X(t))$ and $X(t)$

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can be written in terms of $Z(t)$ through Eq. (7).[7] Namely it is enough, hereafter, to consider a diffusion process $X=X(t)$ with constant diffusion coefficient

$$dX(t)=a(X(t))\,dt+dB(t). \tag{8}$$

We start with the Fokker–Planck equation

$$\frac{\partial}{\partial t}p(x,t|y,u)=-\frac{\partial}{\partial x}[a(x)p(x,t|y,u)]+\frac{\partial^2}{\partial x^2}p(x,t|y,u) \tag{9}$$

equivalent to the stochastic differential equation (8). By the substitution

$$p(x,t|y,u)=q(x,t|y,u)\exp\{-\tfrac{1}{2}[A(x)-A(y)]\} \tag{10}$$

with $A(x)=-\int^x a(x')\,dx'$, the Fokker–Planck equation (9) can be transformed into a self-adjoint form[8,9]

$$\frac{\partial}{\partial t}q(x,t|y,u)=\left(\frac{\partial^2}{\partial x^2}+V(x)\right)q(x,t|y,u), \tag{11}$$

where the "potential" function $V(x)$ is defined as

$$\begin{aligned}V(x)&=-\tfrac{1}{4}A'(x)^2+\tfrac{1}{2}A''(x)\\&=-\tfrac{1}{4}a(x)^2-\tfrac{1}{2}a'(x).\end{aligned} \tag{12}$$

The Feynman–Kac formula[6] asserts that an elementary solution of Eq. (11) is given by

$$q(x,t|y,u)=\int_{\Omega\binom{x|y}{t|u}}\exp\left(\int_u^t V(\gamma(s))\,ds\right)\mu(d\gamma), \tag{13}$$

where μ denotes a Wiener measure with diffusion constant equal to unity. Correspondingly an elementary solution of the Fokker–Planck equation (9) can be represented in terms of integration over continuous sample paths

$$\begin{aligned}p(x,t|y,u)&=\exp\{-\tfrac{1}{2}[A(x)-A(y)]\}\times\int_{\Omega\binom{x|y}{t|u}}\exp\left(\int_u^t V(\gamma(s))\,ds\right)\mu(d\gamma)\\&=\int_{\Omega\binom{x|y}{t|u}}\exp\left(-\tfrac{1}{2}\int_u^t dA(\gamma(s))+\int_u^t V(\gamma(s))\,ds\right)\mu(d\gamma),\end{aligned} \tag{14}$$

because we have

$$\int_u^t dA(\gamma(s))=A(x)-A(y) \tag{15}$$

with probability one.

The mathematical techniques explained above [Eqs. (9)–(14)] are essentially the same as those of Benes and Shepp.[10] In their article emphasis was put on the derivation of the Feynman–Kac formula from the Prokhorov formula[11] making use of transformation (10). In the present paper, on the contrary, emphasis is put on the derivation of a special case of the Prokhorov formula, which is relevant to the Onsager–Machlup formula, in utilizing the Feynman–Kac formula and transformation (10).

Before we proceed with further manipulations with Eq. (14) it is worthwhile to notice two types of stochastic calculus, that is, Itô calculus and Fisk–Stratonovich calculus.[7] Namely the stochastic chain rule in the Itô sense yields

$$dA(\gamma(s))=A'(\gamma(s))\,d\gamma(s)+A''(\gamma(s))\,ds \tag{16}$$

and the one in the Fisk–Stratonovich sense yields

$$dA(\gamma(s))=A'(\gamma(s))\circ d\gamma(s), \tag{17}$$

where $A'(\gamma(s))\,d\gamma(s)$ and $A'(\gamma(s))\circ d\gamma(s)$ are the Itô integral and the Fisk–Stratonovich integral, respectively.

Finally we find that Eq. (14) can be written

$$\begin{aligned}p(x,t|y,u)=&\int_{\Omega\binom{x|y}{t|u}}\exp\left(-\tfrac{1}{2}\int_u^t A'(\gamma(s))\,d\gamma(s)-\tfrac{1}{4}\int_u^t A'(\gamma(s))^2\,ds\right)\mu(d\gamma)\\=&\int_{\Omega\binom{x|y}{t|u}}\exp\left(\tfrac{1}{2}\int_u^t a(\gamma(s))\,d\gamma(s)-\tfrac{1}{4}\int_u^t a(\gamma(s))^2\,ds\right)\mu(d\gamma)\end{aligned} \tag{18}$$

in the Itô calculus and

$$\begin{aligned}p(x,t|y,u)&\\=&\int_{\Omega\binom{x|y}{t|u}}\exp\left(-\tfrac{1}{2}\int_u^t A'(\gamma(s))\circ d\gamma(s)-\tfrac{1}{4}\int_u^t A'(\gamma(s))^2\,ds+\tfrac{1}{2}\int_u^t A''(\gamma(s))\,ds\right)\mu(d\gamma)\\=&\int_{\Omega\binom{x|y}{t|u}}\exp\left(\tfrac{1}{2}\int_u^t a(\gamma(s))\circ d\gamma(s)-\tfrac{1}{4}\int_u^t a(\gamma(s))^2\,ds-\tfrac{1}{2}\int_u^t a'(\gamma(s))\,ds\right)\mu(d\gamma)\end{aligned} \tag{19}$$

in the Fisk–Stratonovich calculus. Equations (18) and (19) are rigorous expressions of the Onsager–Machlup formula for a one-dimensional nonlinear diffusion process $X=X(t)$. Moreover those expressions seem to present a simple and special demonstration of the Prokhorov formula[11] which provides a Radon–Nikodym derivative of the path probability measure of the process $X=X(t)$ with respect to the Wiener measure μ.

It seems worth noticing that the inverse problem, that is, to renormalize such a Radon–Nikodym derivative into a path probability measure, has been fully investigated by Ezawa, Klauder, and Shepp.[12]

III. MOST PROBABLE PATH

In the preceding section we have derived rigorous expressions of the Onsager–Machlup formula for a one-dimensional nonlinear diffusion process, Eqs. (18) and (19). Although there remains no ambiguity in those expressions, it does not seem clear that they are generalizations of the original Onsager—Machlup formula for a linear diffusion process. Therefore, more familiar but approximative expressions of the Onsager—Machlup formula seems to be needed,

We show, in the present section, that our expression of the Onsager—Machlup formula in the Fisk—Stratonovich calculus Eq. (19) reproduces an approximative expression which may be a direct generalization of the original one obtained by Onsager and Machlup.[2]

It is convenient, following Onsager and Machlup,[2] to approximate the Wiener measure μ in Eq. (19) by an n-gate cylinder measure[1,6]

$$\mu_n(d\gamma)=\frac{\exp[-(x-\gamma_n)^2/4(t-t_n)]}{[4\pi(t-t_n)]^{1/2}}\cdots \frac{\exp[-(\gamma_1-y)^2/4(t_1-u)]}{[4\pi(t_1-u)]^{1/2}}\,d\gamma_n\cdots d\gamma_1 \qquad (20)$$

with $t>t_n>\cdots>t_1>u$. Namely, the transition probability density $p(x,t|y,u)$ of the one-dimensional nonlinear diffusion process $X=X(t)$ has an approximative expression

$$\begin{aligned}p(x,t|y,u)&\propto\int\exp[-(x-\gamma_n)^2/4(t-t_n)]\cdots\exp[-(\gamma_1-y)^2/4(t_1-u)]\\&\times\exp\left[\frac{1}{2}a\left(\frac{x+\gamma_n}{2}\right)(x-\gamma_n)+\cdots+\frac{1}{2}a\left(\frac{\gamma_1+y}{2}\right)(\gamma_1-y)\right]\\&\times\exp\left\{\left[-\frac{1}{4}a\left(\frac{x+\gamma_n}{2}\right)^2-\frac{1}{2}a'\left(\frac{x+\gamma_n}{2}\right)\right](t-t_n)+\cdots\right.\\&\left.+\left[-\frac{1}{4}a\left(\frac{\gamma_1+y}{2}\right)^2-\frac{1}{2}a'\left(\frac{\gamma_1+y}{2}\right)\right](t_1-u)\right\}d\gamma_n\cdots d\gamma_1,\end{aligned} \qquad (21)$$

where we have utilized Eqs. (19) and (20) and the definition of the Fisk- Stratonovich integral.

Now what is left for us is to replace each integration in Eq. (21) by taking the maximum value in the exponent;

$$\begin{aligned}p(x,t|y,u)&\propto\exp\left\{\frac{1}{4}\left(\frac{(x-\bar\gamma_n)^2}{t-t_n}+\cdots+\frac{(\bar\gamma_1-y)^2}{t_1-u}\right)\right.\\&\left.+\frac{1}{2}\left[a\left(\frac{x+\bar\gamma_n}{2}\right)(x-\bar\gamma_n)+\cdots+a\left(\frac{\bar\gamma_1+y}{2}\right)(\bar\gamma_1-y)\right]\right\}_{\max}\\&\times\exp\left\{\left[-\frac{1}{4}a\left(\frac{x+\bar\gamma_n}{2}\right)^2-\frac{1}{2}a'\left(\frac{x+\bar\gamma_n}{2}\right)\right](t-t_n)+\cdots\right.\\&\left.+\left[-\frac{1}{4}a\left(\frac{\bar\gamma_1+y}{2}\right)^2-\frac{1}{2}a'\left(\frac{\bar\gamma_1+y}{2}\right)\right](t_1-u)\right\}_{\max}\\&=\exp\left\{\left[-\frac{1}{4}\left(\frac{x-\bar\gamma_n}{t-t_n}\right)^2+\frac{1}{2}a\left(\frac{x+\bar\gamma_n}{2}\right)\cdot\frac{x-\bar\gamma_n}{t-t_n}-\frac{1}{4}a\left(\frac{x+\bar\gamma_n}{2}\right)^2\right.\right.\\&\left.-\frac{1}{2}a'\left(\frac{x+\bar\gamma_n}{2}\right)\right](t-t_n)+\cdots+\left[-\frac{1}{4}\left(\frac{\bar\gamma_1-y}{t_1-u}\right)^2+\frac{1}{2}a\left(\frac{\bar\gamma_1+y}{2}\right)\right.\\&\left.\left.\cdot\frac{\bar\gamma_1-y}{t_1-u}-\frac{1}{4}a\left(\frac{\bar\gamma_1+y}{2}\right)^2-\frac{1}{2}a'\left(\frac{\bar\gamma_1+y}{2}\right)\right](t_1-u)\right\}_{\max}\\&=\exp\left(\left\{-\frac{1}{4}\left[\left(\frac{x-\bar\gamma_n}{t-t_n}\right)-a\left(\frac{x+\bar\gamma_n}{2}\right)\right]^2-\frac{1}{2}a'\left(\frac{x+\bar\gamma_n}{2}\right)\right\}(t-t_n)\right.\\&+\cdots+\left\{-\frac{1}{4}\left[\left(\frac{\bar\gamma_1-y}{t_1-u}\right)-a\left(\frac{\bar\gamma_1+y}{2}\right)\right]^2-\frac{1}{2}a'\left(\frac{\bar\gamma_1+y}{2}\right)\right\}\\&\left.\times(t_1-u)\right)_{\max},\end{aligned} \qquad (22)$$

where $\bar\gamma_j$ denotes the most probable value of $X(t_j)$ for $j=1,2,\cdots,n$.

Passing to the limit $n\to\infty$ we finally obtain an approximative expression of the Onsager—Machlup formula for the one-dimensional nonlinear diffusion process $X=X(t)$,

$$\begin{aligned}p(x,t|y,u)&\propto\exp\Big(-\tfrac{1}{4}\int_u^t[\dot{\bar\gamma}(s)-a(\bar\gamma(s))]^2ds\\&\quad-\tfrac{1}{2}\int_u^t a'(\bar\gamma(s))\,ds\Big)_{\max}\\&=\exp\Big[-\int_u^t L_{\mathrm{OM}}(\dot{\bar\gamma}(s),\bar\gamma(s))\,ds\Big]_{\max},\end{aligned} \qquad (23)$$

where L_{OM} is the Onsager—Machlup Lagrangian for the process $X=X(t)$ defined as

$$\begin{aligned}&L_{\mathrm{OM}}(\dot{\bar\gamma}(s),\bar\gamma(s))\\&=\tfrac{1}{4}[\dot{\bar\gamma}(s)-a(\bar\gamma(s))]^2+\tfrac{1}{2}a'(\bar\gamma(s))\end{aligned} \qquad (24)$$

Note that an equation of motion satisfied by the most probable path $\bar\gamma=\bar\gamma(t)$ is derived by a variational problem for the Onsager—Machlup Lagrangian

$$\int_u^t L_{\mathrm{OM}}(\dot{\bar\gamma}(s),\bar\gamma(s))\,ds=\min, \qquad (25)$$

that is, the Euler—Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L_{\mathrm{OM}}}{\partial\dot{\bar\gamma}}\right)-\frac{\partial L_{\mathrm{OM}}}{\partial\bar\gamma}=0 \qquad (26)$$

provides the most probable path $\bar\gamma=\bar\gamma(t)$ for the one-dimensional nonlinear diffusion process $X=X(t)$.

An equation of motion thus obtained from Eq. (26) with Eq. (24),

$$\ddot{\bar\gamma}(t)-a(\bar\gamma(t))a'(\bar\gamma(t))-a''(\bar\gamma(t))=0, \qquad (27)$$

extends Onsager and Machlup's result[2] to the case of a one-dimensional nonlinear diffusion process.

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Recoupling coefficients of the symmetric group involving outer plethysms

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In a series of papers we have examined the properties of certain double coset matrix elements (DCME) in the representations of the symmetric group S_N that act as recoupling coefficients for outer products carried out via alternate subgroup sequences. In this paper we examine these same properties using symmetrized outer products in S_N, which are also known as outer plethysms. The notions of double coset representative, symbol, and matrix element are extended to this case using the theory of semidirect products and little groups. The recoupling coefficients between bases symmetry adapted with respect to the usual outer product and the outer plethysm are examined in detail. Because of the Weyl–Schur construction of irreducible tensors, the recoupling theory of S_N is central to a unified recoupling theory of the general linear group and its subgroups.

I. INTRODUCTION

In a series of articles[1] we have been attempting to develop a theory of recoupling coefficients (Racah algebra) applicable to the integral representations of the general linear group of arbitrary dimension $\mathrm{Gl}(d)$, and by direct extension to its unitary $\mathrm{U}(d)$ and unitary unimodular $\mathrm{SU}(d)$ subgroups. The distinguishing feature of this development is that the role of the symmetric group S_N in designating the coupling scheme is always explicitly displayed. Extension of this development to the basis labeling scheme used in spectroscopic shell theory (see the following paper) has required a more thorough analysis of symmetrized outer products in S_N. The operation of symmetrizing an outer product is termed by Littlewood[2] an outer plethysm and is treated in some detail in his book and in a book by Wybourne.[3]

A general thesis of our development of the Racah algebra of $\mathrm{Gl}(d)$ is that the underlying vector space of whatever dimension simply acts as a carrier space of matrix representations of S_N. The nontrivial aspects of the general recoupling algebra are in one to one correspondence with the algebraic structure in S_N. Thus our development can be separated into consideration of the recoupling algebra of S_N as in this paper, and consideration of the correspondence of this algebra with the Racah algebra of $\mathrm{Gl}(d)$ as in the following paper.

The theory of recoupling coefficients developed so far has been based on a double coset (DC) decomposition $\otimes S_{{}_iN}\backslash S_N/\otimes S_{N_j}$, ${}_iN\vdash N$, $N_j \vdash N$ ($\vdash$ meaning a partition of). The recoupling coefficients have been identified with double coset matrix elements (DCME) of S_N in bases symmetry adapted to the DC decomposition. Weighted double coset matrix elements (WDCME) satisfy orthogonality and completeness conditions in S_N. Certain of these WDCME have been identified with the isoscalar factor of the Racah algebra. In this paper we extend the DC development of S_N to consider the process of outer plethysm applicable when two or more of the parts ${}_iN$ or N_j become equal. Designation of the DC and the DCME requires the use of semidirect group products and the representation theory of little groups.[4] The designation, let along the evaluation of a general DCME in this scheme is involved. Fortunately it is the simpler DCME (not to say trivial) that are required for extending the development to the recoupling coefficients of shell theory.

In Sec. II we review the general theory of DC decomposition and the pertinent results as applied to S_N. Semidirect products and the representation theory of little groups are reviewed in Sec. III. The extension to outer plethysm is considered in Sec. IV. This is followed by some brief comments on phase conventions in the final section.

II. NOTATION AND RESUME OF PREVIOUS RESULTS

Any group G can be expressed as a sum of disjoint double cosets $H\backslash G/K$ with respect to any subgroups H and K (perhaps identical),

$$G=\bigcup_q HqK, \quad q \text{ a double coset representative (DCR).}$$

A matrix representation of G can be considered that is symmetry adapted to (different or identical) subgroup sequences on the left (lower) and the right (upper) indices

$$G \nearrow K \to q^{-1}Hq \cap K \equiv {}^qL,$$
$$G \searrow H \to H \cap qKq^{-1} \equiv L_q.$$

Note this is a representation in the normal sense only for $H\equiv K$. The intertwining subgroups ${}^qL$ and L_q are isomorphic under conjugation by q. Because $q\,{}^qL = L_q q$, Schur's lemma requires the intertwining matrix assigned to the DCR to have double coset matrix elements of the form

$$\begin{bmatrix} \lambda & q \\ {}_i\lambda\ {}_i\lambda_j'\ m' & \lambda_j\ {}_i\lambda_j\ m \end{bmatrix} = \delta_{{}_i\lambda_j' {}_i\lambda_j}\delta_{m'm}\begin{bmatrix} \lambda & \lambda_j \\ {}_i\lambda & {}_i\lambda_j \end{bmatrix}, \qquad (2.1)$$

where λ, ${}_i\lambda$, λ_j, ${}_i\lambda_j$ label the irreps of the corresponding groups, G, H, K, and ${}^qL\approx L_q$ respectively. The (assumed) unitarity of the matrix representation requires unitarity of the DCME, the pair $({}_i\lambda,\lambda_j)$ designating the rows and columns with λ and ${}_i\lambda_j$ fixed.

The group completeness condition can be regarded as establishing an isometry between inducing from a subgroup H by action of the coset factors G/H, and inducing by action of matrix basis projectors $(\lambda\,|\,m\,i)$

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symmetry adapted to the subgroup. The isometry is a specification of the duality between inducing and restricting as expressed in Frobenius' reciprocity relation. With matrix basis projectors defined as

$$(\lambda\,|\,mn) \equiv \frac{|\lambda|}{|G|}\sum_g \begin{bmatrix} \lambda & g^{-1} \\ & \\ n & m \end{bmatrix} g = (\lambda\,|\,nm)^*$$

the orthogonality relations become

$$(\lambda'\,|\,m'n')^*(\lambda\,|\,mn) = \delta^{\lambda\lambda'}\delta_{mm'}(\lambda\,|\,n'\,n)$$

and the completeness relation becomes

$$\sum_{\lambda mn}\begin{bmatrix} \lambda & g \\ & \\ m & n \end{bmatrix}(\lambda\,|\,mn) = g.$$

When the index $n \to n\lambda_j m_j$ is symmetry adapted to a subgroup H, the orthogonality and completeness relations factor requiring for coset representatives, $\sigma \in G/H$;

$$\sum_{\sigma, m_j}\frac{|\lambda|}{|G|}\begin{bmatrix} \lambda' & \sigma \\ & \\ m' & n'\lambda_j m_j \end{bmatrix}^*\begin{bmatrix} \lambda & \sigma \\ & \\ m & n\lambda_j m_j \end{bmatrix} = \delta^{\lambda\lambda'}\delta_{mm'}\delta_{nn'}\frac{|\lambda_j|}{|H|}$$

and

$$\sum_{\lambda mn}\frac{|\lambda|}{|G|}\begin{bmatrix} \lambda & \sigma' \\ & \\ m & n\lambda_j m_j' \end{bmatrix}^*\begin{bmatrix} \lambda & \sigma \\ & \\ m & n\lambda_j m_j \end{bmatrix} = \delta_{\sigma\sigma'}\frac{|\lambda_j|}{|H|}\delta_{m_j m_j'}.$$

Extension of this factorization to double coset representatives $K\backslash G/H$ leads to the unitarity of the WDCME,

$$\left\{\frac{|H|\;|K|\;|\lambda|\;|_i\lambda_j|}{|G|\;|L||_i\lambda|\;|\lambda_j|}\right\}^{1/2}\begin{bmatrix} \lambda & \lambda_j \\ & \\ _i\lambda & _i\lambda_j \end{bmatrix}, \tag{2.2}$$

the pair $(\lambda,\ q\ _i\lambda_j)$ designating the (square) rows and columns with $_i\lambda$, λ_j fixed. In (2.2) $|G|$, $|H|$, $|K|$, and $|L|$ denote the order of the groups G, H, K, and L_q respectively and $|\lambda|$ denotes the dimension of the corresponding irreps. It has further been shown 1(c) that certain specific applications of the unitarity expressed in (2.2) give well-known dimensional results in the theory of induced representations in accord with Frobenius' reciprocity relation and Mackey's subgroup theorem. More exactly, Eq. (2.2) expresses the element of the transformation matrix between the two bases shown to be equivalent by Mackey's subgroup theorem.

As applied to the symmetric group with the double coset decomposition $S_{_1N}\otimes S_{_2N}\backslash S_N/S_{N_1}\otimes S_{N_2}$, the DCR are in one to one correspondence with double coset symbols

$$\begin{bmatrix} N & N_1 & N_2 \\ _1N & _1N_1 - K & _1N_2 + K \\ _2N & _2N_1 + K & _2N_2 - K \end{bmatrix}$$

where K takes all integral values such that all entires are nonnegative and

$$S_{_1N}\otimes S_{_2N}\cap S_{N_1}\otimes S_{N_2} = S_{_1N_1}\otimes S_{_1N_2}\otimes S_{_2N_1}\otimes S_{_2N_2}.$$

The DCME can be considered real and take the form

$$\begin{bmatrix} \lambda & \lambda_1 & \lambda_2 \\ _1\lambda & _1\lambda_1 & _1\lambda_2 \\ _2\lambda & _2\lambda_1 & _2\lambda_2 \end{bmatrix}$$

where the rows or columns must couple according to the Littlewood–Richardson rules[2] for outer products of the symmetric group. Multiplicity labels may be required to uniquely specify the outer product coupling. Such labels although supressed in the notion used here are implied in the process of carrying out the Littlewood–Richardson rules.

Extension of the above procedure to subgroups signifying multiproducts $S_N/\otimes S_{N_i}$ requires the additional specification of a series of binary couplings in arbitrary order. This complication can be avoided when the subgroups are over equivalent sets as $S_{nN}/(S_n)^N$. An intermediate subgroup can be introduced, $S_{nN}/S_N \circledS (S_n)^N$, where $\circledS$ indicates a semidirect product, and the theory of little groups can be used to specify the coupling chain. This is the subgroup sequence used in shell theory, and the principal goal of this paper is to extend our double coset considerations to such subgroups.

In the above and what follows we use the notation:

$\otimes$ a direct product over the indicated terms,

$\circledS$ a semidirect product,

$\odot$ an outer plethysm (symmetrized outer product),

Greek letters λ, α, μ label irreps of the various groups.

III. TENSOR PRODUCTS AND THE THEORY OF LITTLE GROUPS

Let G^N be the Nth rank direct product group with elements $(g_1, g_2, \ldots, g_N) \equiv \{g\}$ which is a normal subgroup of the semidirect product group $S_N \circledS G^N$ with elements $(\pi, \{g\})$. The group combination law is

$$(\pi, \{g\})(\pi', \{g'\}) = (\pi\pi', \{g\}\{\pi g'\}), \tag{3.1}$$

where $\{\pi g'\}$ indicates the N-tuple $\{g'\}$ reordered in accordance with the permutation π. Denoting the irreducible representations of G by α_i, the irreps of G^N are designated by sets $\{\alpha_i^{N_i}\}$ where N_i indicates the frequency of the irrep α_i in the direct product. More exactly $\{\alpha_i^{N_i}\}$ labels an orbit of G^N, different irreps corresponding to different orderings of the α_i, but our notation need not make this distinction explicit. The irrep $\{\alpha_i^{N_i}\}$ has invariance (stability or little) group $\otimes\,(S_{N_i}\circledS (G)^{N_i})$ with irreps designated by products of paired labels $\otimes\,(\lambda_i\alpha_i)$ with $\lambda_i \vdash N_i$. In accordance with the theory of little groups these same labels suffice to specify the irreps of the normalizer group which we symbolize by the set $\{\lambda_i\alpha_i\}$. The induction relations are:

$$\{\alpha_i^{N_i}\}\uparrow\otimes\,(S_{N_i}\circledS (G)^{N_i}) = \otimes\,\sum|\lambda_i|\,(\lambda_i\alpha_i) \tag{3.2}$$

for any irrep in the orbit of $\{\alpha_i^{N_i}\}$, and

$$\otimes\,(\lambda_i\alpha_i)\uparrow S_N \circledS (G)^N = \{\lambda_i\alpha_i\},\ \text{an irrep.} \tag{3.3}$$

The irreps $\{\alpha_i^{N_i}\}$, $\otimes\,(\lambda_i\alpha_i)$, and $\{\lambda_i\alpha_i\}$ have dimensions $\otimes|\alpha_i|^{N_i}$, $\otimes\,|\lambda_i|\;|\alpha_i|^{N_i}$, and $\otimes\binom{N}{N_i}|\lambda_i|\;|\alpha_i|^{N_i}$ respectively. $[\binom{N}{N_j}$ is the multinomial $N!/\otimes N_j!.]$ As noted in

Sec. 2 the induction process can be considered in its specifically reduced form using matrix basis projectors. For simplicity we consider induction from the irrep $\{\alpha^N\}$ so that the little group is the normalizer.

The matrix basis projectors for the irrep $\{\lambda\alpha\}$ of $S_N \circledS (G)^N$ can be written as

$$(\lambda|ml) \circledS \{(\alpha|m_i l_i)\} = \frac{|\lambda|}{N!}\,\frac{|\alpha|}{|G|}\sum_{\pi\{g\}} \begin{bmatrix} \lambda & \pi \\ m & l \end{bmatrix} \otimes \begin{bmatrix} \alpha & g_i \\ m_i & l_i \end{bmatrix} \times (\pi, \{\pi g\}). \qquad (3.4)$$

By considering the action of an element $(\pi', \{g'\})$ on the projector one verifies that a carrier space of $S_N \circledS (G)^N$ is specified by the fixed set $\{l_i\}$ and index l with m varying over $|\lambda|$ values and the set $\{m_i\}$ varying over $|\alpha|^N$ value. By the above little group theory it is also an irreducible space. The $|\lambda|$ possible values of the index l specify the multiplicity in the induction $\{\alpha^N\} \uparrow S_N \circledS (G)^N = \sum |\lambda| (\lambda\alpha)$.

This basic construction of tensor product representation classified w.r.t. S_N is used in several different schemes. The Schur–Weyl construction of irreducible tensor spaces takes G to be the general linear group in n dimensions $\mathrm{Gl}(n)$ or one of its subgroups and considers the restriction to the diagonal elements $(g_1, g_2, \ldots, g_N) \to (g, g, \ldots, g)$ and thus labels the spaces w.r.t. the subgroup link

$$\begin{array}{ccc} S_N \circledS (G)^N & \text{——} & (G)^N \\ | & & | \\ S_N \otimes G & \text{——} & G \end{array} .$$

When α is the defining irrep of $\mathrm{Gl}(n)$, the restriction to $S_N \otimes G$ yields the basic Schur–Weyl scheme. If α is a tensor irrep of $\mathrm{Gl}(n)$, the restriction to $S_N \otimes G$ yields the outer plethysm $(\alpha \circledcirc \lambda)$ evaluated in G.

For $G \equiv S_n$ the subgroup link may be extended to

$$\begin{array}{ccccc} S_{nN} & \text{——} & S_N \circledS (S_n)^N & \text{——} & (S_n)^N \\ | & & | & & | \\ S_n \circledS (S_N)^n & \text{——} & S_N \otimes S_n & \text{——} & S_n \end{array} .$$

The restriction of $\{\lambda\alpha\}$ to $S_N \otimes S_n$ is termed by Littlewood an inner plethysm because it is the λth symmetrized power of the irrep α evaluated in S_n. The induction of $\{\lambda\alpha\}$ to S_{nN} is termed by Littlewood an outer plethysm because it is the λth symmetrized power of the irrep α evaluated in S_{nN}. Again using the isometry between induction and projection, the matrix basis projector that realizes the decomposition of the outer plethysm can be written as

$$(\Lambda|ML)\,[(\lambda|ml) \circledS \{(\alpha|m_i l_i)\}]. \qquad (3.5)$$

If the outer plethysm decomposes as $\{\lambda\alpha\} \uparrow S_{nN} = \sum f_\Lambda(\Lambda)$, then only f_Λ different values of L project independent elements in the algebra of S_{nN}. By Frobenius' reciprocity relation, if the basis label L is symmetry adapted to the subgroup $S_N \circledS (S_n)^N$, then the $|\Lambda|$ different projectors for varying L are differentiated into f_Λ and $(|\Lambda| - f_\Lambda)$ projectors which when acting on the space $\{\lambda\alpha\}$ give f_Λ independent spaces or the null result respectively. Because outer plethysms are important to the labeling used in spectroscopic shell theory we examine the DC decompositions pertinent to these subgroups in the next section.

IV. OUTER PLETHYSM AND DOUBLE COSETS

A. Semidirect products and DC symbols

For direct product subgroups $\otimes S_{i^N} \backslash S_N / \otimes S_{N_j}$ the DC

TABLE I. Examples of designating double cosets in various subgroup decompositions.

DC composition	DC symbol characterized by (N-tuple or, array)	Intersection subgroup
(a) $S_{2N-1} \otimes S_1 \backslash S_{2N}/S_2 \circledS (S_N)^2$	$(1, 0)$	$S_{N-1} \otimes S_N$
(b) $S_{2N-1} \otimes S_1 \backslash S_{2N}/S_2 \circledS (S_2)^N$	$(1, 0^{N-1})$	$S_{N-1} \circledS (S_2)^{N-1}$
(c) $S_2 \circledS (S_N)^2 \backslash S_{2N}/S_2 \circledS (S_N)^2$	$(N-k, k)\quad 0 \le k < N/2$ $(N/2)^4\quad k = N/2$	$S_2 \circledS (S_{N-k} \otimes S_k)^2$ $S_2^2 \circledS (S_{N/2})^4$
(d) $S_2 \circledS (S_N)^2 \backslash S_{2N}/S_N \circledS (S_2)^N$	$(2^k, 1^{N-2k}, 0^k)\quad 0 \le k \le N/2$	$S_2 \circledS [S_k \circledS (S_2)^k]^2 \otimes S_{N-2k}$
(e) $S_3 \circledS (S_3)^3 \backslash S_9/S_3 \circledS (S_3)^3$	(a) 300 030 003	(a) $S_3 \circledS (S_3)^3$
	(b) 300 021 012	(b) $S_3 \otimes [S_2 \circledS (S_2)^2]$
	(c) 210 021 102	(c) $C_3 \circledS (S_2)^3$
	(d) 210 012 111	(d) $S_2 \circledS (S_2)^2$
	(e) 111 111 111	(e) $(S_3)^2$
(f) $S_4 \circledS (S_3)^4 \backslash S_{12}/S_3 \circledS (S_4)^3$	(a) 300 030 003 111	(a) $S_3 \circledS (S_3)^3$
	(b) 300 030 102 012	(b) $S_2 \circledS [S_3 \otimes S_2]^2$
	(c) 300 120 012 012	(c) $S_3 \otimes S_2 \otimes S_2 \circledS (S_2)^2$
	(d) 300 021 012 111	(d) $S_3 \otimes S_2 \circledS (S_2)^2$
	(e) 210 210 012 012	(e) $S_2 \circledS [S_2 \circledS (S_2)^2]^2$
	(f) 210 201 022 011	(f) $S_2 \circledS [(S_2)^2]^2$
	(g) 210 021 102 111	(g) $C_3 \circledS (S_2)^3$
	(h) 201 021 111 111	(h) $S_2 \otimes S_2 \circledS (S_2)^2$
	(i) 111 111 111 111	(i) $S_4 \otimes S_3$

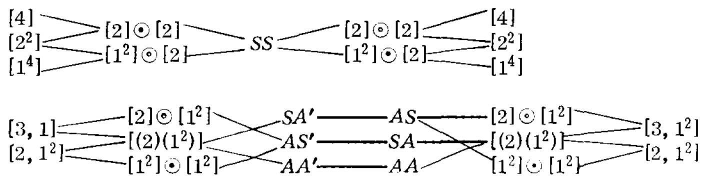
S_N indicates the symmetric group on N items; C_N indicates the cyclic group on N items.

TABLE II. Evaluation of transpose DCME for $S_2 \circledS (S_2)^2\backslash S_4/S_2 \circledS (S_2)^2$.

(a) Character tables

	$S_2 \circledS (S_2)^2$					$S_2\otimes S_2$				
irrep\class	(1^4)	(12) (34)	$\begin{Bmatrix}(13)\ (24)\\(14)\ (23)\end{Bmatrix}$	$2(2,1^2)$	$2(4)$		(1^4)	}(12) (34){	}(13) (24){	}(14) (23){
$[2]\odot[2]$	1	1	1	1	1	SS	1	1	1	1
$[1^2]\odot[2]$	1	1	1	−1	−1					
$[2]\odot[1^2]$	1	1	−1	1	−1	AS	1	1	−1	−1
$[1^2]\odot[1^2]$	1	1	−1	−1	1					
$[(2)(1^2)]$	2	−2	0	0	0	SA	1	−1	1	−1
						AA	1	−1	−1	1

(b) Branching diagrams DCR $q=(23)$



(c) DCME for bipartition irreps

$$\begin{bmatrix} [2^2] & [2]\odot[2] & [1^2]\odot[2] \\ [2]\odot[2] & SS & \\ [1^2]\odot[2] & & \end{bmatrix} = \begin{pmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{bmatrix} [3,1] & [2]\odot[1^2] & \multicolumn{2}{c}{[(2)(1^2)]} \\ [2]\odot[1^2] & 0 & \overbrace{AS' \sim SA} & 0 \\ [(2)(1^2)] & \begin{cases} SA' \sim AS \\ 0 \end{cases} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ AA \end{matrix} \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In Tables II and III the irreps $\{\lambda\alpha\}$ of $S_N \circledS (S_N)^N$ are simply designated as $[\alpha]\odot[\lambda]$. The outer plethysm is then given by the compatability relations with S_{nN}.

are designated by DC symbols which are simply a rectangular array of nonnegative integers ${}_iN_j$ with $\sum_i {}_iN_j = N_j$ and $\sum_j {}_iN_j = {}_iN$. The intertwining subgroup is isomorphic to the direct product $\otimes S_{{}_iN_j}$. The DC symbols for $(S_{n'})^{N'}\backslash S_{nN}/(S_n)^N$ with $n'N' = nN$ are simply N' by N arrays of positive integers ${}_in_j$ different orderings corresponding to different DC. The effect of introducing the normalizers as subgroups $S_{N'} \circledS (S_{n'})^{N'}\backslash S_{nN}/S_N \circledS (S_n)^N$ is to coalesce all arrays $\{{}_in_j\}$ related to each other by permutations of the rows and columns into a single DC. That is the DC symbol can be considered to be an ordered N' by N array of nonnegative integers ${}_in_j$ with $\sum_i {}_in_j = n$ and $\sum_j {}_in_j = n'$. The intertwining subgroup is isomorphic to $C \circledS [\otimes S_{{}_in_j}]$ where C is the permutation subgroup $C \subseteq S_{N'} \otimes S_N$ that leaves the array $\{{}_in_j\}$ invariant. A particularly simple example is the decomposition $S_{1^n} \otimes S_{2^n}\backslash S_{nN}/S_N \circledS (S_n)^N$ with the DC symbols given by ordered partitions of ${}_2\eta = \sum kw_k$, $0 \leq k \leq n$. The DC symbol is thus equivalent to the specification of a set of $n+1$ weights $\{w_k\}$. The intertwining subgroup is isomorphic to $\otimes [S_{w_k} \circledS [S_{n-k} \otimes S_k]^{w_k}]$. Table I gives several examples of DC decompositions using semidirect products. The order of the various subgroups in Table I can be used to check the normalization of Eq. (2.2) with $\lambda = \lambda^0$ the identity irrep.

B. Outer plethysms and DCME

The DCME have irrep labels corresponding to their respective groups. In particular the irreps of semidirect product groups are labeled in the manner discussed above. The orthogonality relations with the group order, irrep labels, and dimensions correspondingly modified are directly applicable. The DC are real and those related by association w.r.t. induction from the alternating group are equal up to a phase. Although semidirect products substantially reduce the number of DC to be considered, one must be careful to correctly label equivalent irreps of the isomorphic intertwining subgroups.

For example in the DC decomposition $S_{1^n} \otimes S_{2^n}\backslash S_{nN}/S_N \circledS (S_n)^N$ the equivalent intertwining subgroups on the left and the right are

$$\otimes [S_{w_k} \circledS (S_{n-k})^{w_k}] \otimes [S_{w_k} \circledS (S_k)^{w_k}]$$

$$\approx \otimes [S_{w_k} \circledS [S_{n-k} \otimes S_k]^{w_k}], \tag{4.1}$$

where $\{w_k\} \vdash N$ and $\{kw_k\} \vdash {}_2\eta$. The elements of S_{w_k} on both sides are identical and labeling the equivalent irreps of the intersection subgroup requires the use of the plethysm product rule (Ref. 3, p. 52)

$$\sum_{\mu,\nu} g_{\lambda\mu\nu} (A \circledcirc (\mu))(B \circledcirc (\nu)) = (AB \circledcirc (\lambda)), \tag{4.2}$$

where $g_{\lambda\mu\nu}$ is the inner product multiplicity in the symmetric group.

Fortunately the DCME required by shell theory do not require such attention to detail. Two DCME are of particular interest for coupling within equivalent shells. One is with all irreps of S_n being the one-dimensional totally symmetric irrep $(n) \equiv (2j)$, $\lambda \vdash N$ being limited to parts (length of rows) no greater than two or four for

TABLE III. Evaluation of transpose DCME for $S_2 \circledS (S_3)^2 \backslash S_6 / S_2 \circledS (S_3)^2$.

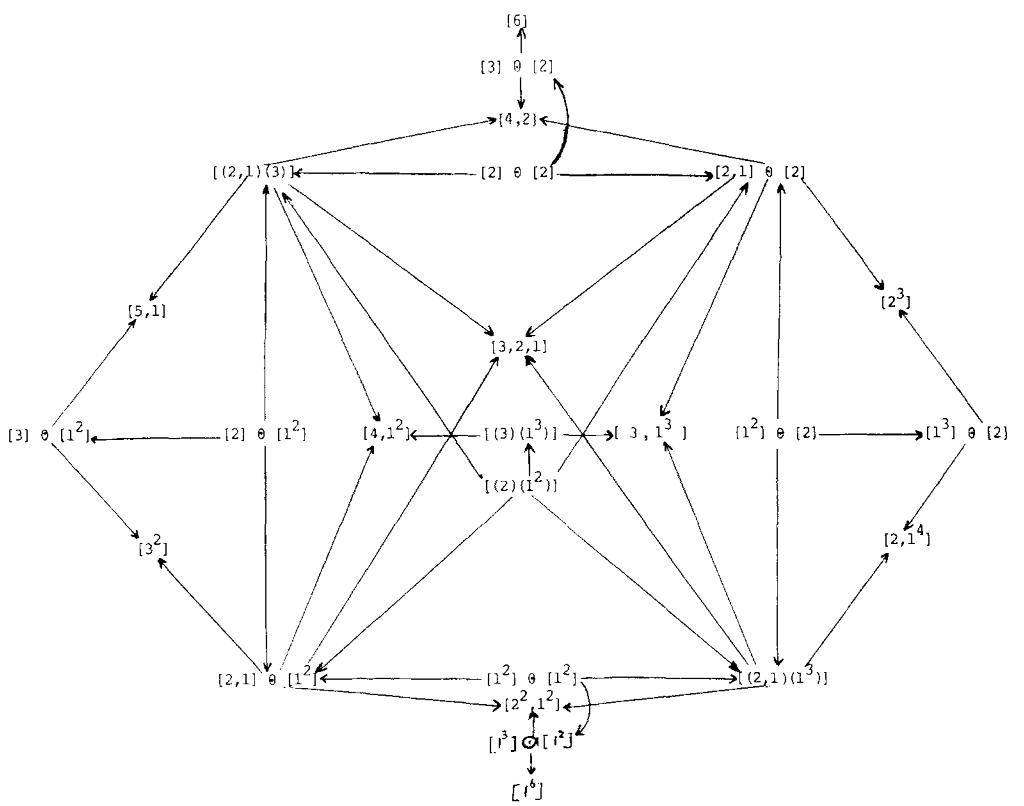
(a) Character tables $S_2 \circledS (S_3)^2$

irrep\class	$1(1^6)$	$6(2,1^4)$	$4(3,1^3)$	$9(2^2,1^2)$	$12(3,2,1)$	$4(3^2)$	$6(2^3)$	$18(4,2)$	$12(6)$
$[3]\odot[2]$	1	1	1	1	1	1	1	1	1
$[3]\odot[1^2]$	1	1	1	1	1	1	−1	−1	−1
$[1^3]\odot[2]$	1	−1	1	1	−1	1	1	−1	1
$[1^3]\odot[1^2]$	1	1	1	1	−1	1	−1	−1	1
$[2,1]\odot[2]$	4	0	1	0	0	−2	2	0	−1
$[2,1]\odot[1^2]$	4	0	1	0	0	−2	−2	0	1
$[(3)(1^3)]$	2	0	2	−2	0	2	0	0	0
$[(2,1)(3)]$	4	2	−2	0	−1	1	0	0	0
$[(2,1)(1^3)]$	4	−2	−2	0	1	1	0	0	0

$S_2 \circledS (S_2)^2$

	(1^6)	$1(2,1^4)$	$1(2^2,1^2)$	$2(2^3)$	$2(4,2)$
$[2]\odot[2]$	1	1	1	1	1
$[2]\odot[1^2]$	1	1	1	−1	−1
$[1^2]\odot[2]$	1	−1	1	1	−1
$[1^2]\odot[1^2]$	1	−1	1	−1	1
$[(2)(1^2)]$	2	0	−2	0	0

(b) Branching diagram (arrows indicate ascent in symmetry and help distinguish irreps of different groups).



(c) DCME for bipartition irreps

$$\begin{bmatrix}[4,2] & [2,1]\odot[2] \\ [2,1]\odot[2] & [1^2]\odot[2]\end{bmatrix} = 1 = \begin{bmatrix}[4,2] & [(2,1)(3)] \\ [(2,1)(3)] & [2]\odot[1^2]\end{bmatrix} = \begin{bmatrix}[3^2] & [2,1]\odot[1^2] \\ [2,1]\odot[1^2] & [(2)(1^2)]\end{bmatrix}$$

$$= -\begin{bmatrix}[3^2] & [2,1]\odot[1^2] \\ [2,1]\odot[1^2] & [1^2]\odot[1^2]\end{bmatrix}, \begin{bmatrix}[5,1] & [3]\odot[1^2] \quad [(2,1)(3)] \\ [3]\odot[1^2] & \\ [(2,1)(3)] & [2]\odot[1^2]\end{bmatrix}$$

$$= \begin{pmatrix}\frac{1}{3} & (\frac{8}{9})^{1/2} \\ (\frac{8}{9})^{1/2} & -\frac{1}{3}\end{pmatrix} = \begin{bmatrix}[3^2] & [3]\odot[1^2] \quad [2,1]\odot[1^2] \\ [3]\odot[1^2] & \\ [2,1]\odot[1^2] & [2]\odot[1^2]\end{bmatrix} = \begin{bmatrix}[4,2] & [2,1]\odot[2] \quad [(2,1)(3)] \\ [2,1]\odot[2] & [(2)(1^2)] \\ [(2,1)(3)] & \end{bmatrix},$$

$$\begin{bmatrix} [4,2] & [3]\odot[2] & [2,1]\odot[2] & [(2,1)(3)] \\ [3]\odot[2] & & & \\ [2,1]\odot[2] & & [2]\odot[2] & \\ [(2,1)(3)] & & & \end{bmatrix} = \begin{pmatrix} -\frac{1}{9} & (\frac{40}{81})^{1/2} & (\frac{40}{81})^{1/2} \\ (\frac{40}{81})^{1/2} & \frac{5}{9} & -\frac{4}{9} \\ (\frac{40}{81})^{1/2} & -\frac{4}{9} & \frac{5}{9} \end{pmatrix}.$$

The full 9 by 9 matrix representation of the tranpose DCR in the irrep [4, 2] is

row or column label $\begin{cases} [3]\odot[2] \\ [2]\odot[2] \end{cases}$ $\begin{matrix} [2,1]\odot[2] \\ [2]\odot[2] \end{matrix}$ $\begin{matrix} [(2,1)(3)] \\ [2]\odot[2] \end{matrix}$ $\begin{matrix} [(2,1)(3)] \\ [2]\odot[1^2] \end{matrix}$ $\begin{matrix} [(2,1)(3)] \\ [(2)(1^2)] \end{matrix}$ $\begin{matrix} [2,1]\odot[2] \\ [(2)(1^2)] \end{matrix}$ $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$ $\begin{matrix} [2,1]\odot[2] \\ [1^2]\odot[2] \end{matrix}$

$$\begin{pmatrix} -\frac{1}{9} & (\frac{40}{81})^{1/2} & (\frac{40}{81})^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 \\ (\frac{40}{81})^{1/2} & \frac{5}{9} & -\frac{4}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ (\frac{40}{81})^{1/2} & -\frac{4}{9} & \frac{5}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & (\frac{8}{9})^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & (\frac{8}{9})^{1/2} & 0 \\ 0 & 0 & 0 & 0 & (\frac{8}{9})^{1/2} & 0 & +\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\frac{8}{9})^{1/2} & 0 & +\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) Recoupling coefficient matrix of SU(2) for $\begin{bmatrix} 1 & j_1 & j_2 \\ {}_1j & {}_1j_1 & \frac{1}{2} \\ {}_2j & \frac{1}{2} & {}_2j_2 \end{bmatrix}$

row or column label $\begin{cases} \begin{bmatrix} j_1 & j_2 \\ {}_1j_1 & {}_2j_2 \end{bmatrix} \end{cases}$ $\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$$\begin{matrix} -\frac{1}{9} & (\frac{40}{81})^{1/2} & -(\frac{20}{81})^{1/2} & -(\frac{20}{81})^{1/2} & 0 & 0 & 0 & 0 & 0 \\ (\frac{40}{81})^{1/2} & \frac{5}{9} & (\frac{8}{81})^{1/2} & (\frac{8}{81})^{1/2} & 0 & 0 & 0 & 0 & 0 \\ -(\frac{20}{81})^{1/2} & (\frac{8}{81})^{1/2} & \frac{7}{9} & -\frac{2}{9} & 0 & 0 & 0 & 0 & 0 \\ -(\frac{20}{81})^{1/2} & (\frac{8}{81})^{1/2} & -\frac{2}{9} & \frac{7}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & (\frac{8}{9})^{1/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & (\frac{8}{9})^{1/2} & 0 \\ 0 & 0 & 0 & 0 & (\frac{8}{9}) & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\frac{8}{9})^{1/2} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

the atomic or nuclear cases, Λ being limited to bipartitions $[jN+J,\ jN-J]$, and ${}_i\Lambda$ being limited to the totally symmetric irreps $[jN+M]$ and $[jN-M]$. The DCME then have the form

$$\begin{bmatrix} [jN+J,\ jN-J] & [2j]\odot[\lambda] \\ [jN+M] & \otimes([2j-k]\odot[w_k]) \\ [jN-M] & \otimes([k]\odot[w_k]) \end{bmatrix}. \qquad (4.3)$$

Such DCME are considered in Eq. (5.3) and in Table I (in their WDCME form) of the following paper. A second case of use in shell theory has ${}_1\eta = n_1N$, ${}_2\eta = n_2N$ and $w_0 = {}_1N$, $w_n = {}_2N$. These DCME have the form

$$\begin{bmatrix} \Lambda & \alpha\odot\lambda \\ {}_1\Lambda & \alpha\odot{}_1\lambda \\ {}_2\Lambda & \alpha\odot{}_2\lambda \end{bmatrix} \qquad (4.4)$$

and are considered in Eq. (5.5) and Table II of the following paper. In either case the inner and plethysm product rules are trivial and these DCME are not directly complicated by the nuances discussed in the preceding paragraph.

The orthogonality relations allow the evaluation of the DCME up to a sign in the case of only two DC [as in (a), (b), and (c) for $N<4$ of Table I]. As an example we consider $N=2$, and 3 for case (c) of Table I. There are only two DC, one DC representative being the identity element with unit diagonal matrix representation in both cases. The other DC representative may be taken as a transpose with a self transpose matrix representation. The DCME for Λ a bipartition representation are evaluated in Tables II ($N=2$) and III ($N=3$). One must first establish the appropriate labels for nonvanishing DCME. Character tables and compatability relations are given in parts (a), and (b) of these tables. Any closed loop completing the subgroup linkage starting from a given irrep Λ (of S_4 or S_6) establishes the labels of a (possibly) nonvanishing DCME. The DCME evaluated via the orthogonality relations are given in part (c) of these tables. Orthogonality of the matrices is obvious. The interested reader is urged to check the orthogonality relations involving the WDCME. The DCME for irrep [3,1] in Table II(c) exemplifies some of the care that must be exercised in considering the isomorphism between the intertwining groups. Conjugation by the transpose $q=(23)$ interchanges the irreps AS and SA of $S_2 \otimes S_2$ and results in the permutation matrix listed.

Because of the identification of DCME of S_N and recoupling coefficients of Gl(n), the values of DCME with bipartition irreps Λ can be checked with corresponding recoupling coefficients of SU(2). For example, the 9 by 9 matrix representing the transpose DCR in the irrep [4,2] will have elements labeled in SU(2) by the angular momentum recoupling coefficients of the form

$$\begin{bmatrix} 1 & j_1 & j_2 \\ {}_1j & {}_1j_1 & \frac{1}{2} \\ {}_2j & \frac{1}{2} & {}_2j_2 \end{bmatrix},$$

where ${}_1j$, ${}_2j$, j_1, j_2 have values $\frac{1}{2}$ or $\frac{3}{2}$ and ${}_1j_1$, ${}_2j_2$ have values 1 or 0. Using well established procedures for SU(2), the 9 by 9 recoupling matrix evaluates as shown in Table III(d). A symmetrizing operation on the third and fourth rows and columns corresponding to conjugation by the matrix

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

must be performed to obtain the states labeled by $[(2,1)(3)]$, and $[2]\odot[2]$ or $[2]\odot[1^2]$. This leads to the matrix of Table III(c) up to a sign. The phase has not been adjusted because we wish to emphasize that while certain relative phases are fixed by the group algebra (the characteristics of the two matrices are the same), a consistent phase convention requires consideration of both the groups S_N and Gl(n).

The identification of DCME in S_N with recoupling coefficients in SU(n) relates different DCME in S_N via their equivalence in SU(n). E.g.

$$\begin{bmatrix} [4,2] & [3] & [2,1] \\ [3] & [2] & [1] \\ [2,1] & [1] & [1^2] \end{bmatrix} = \begin{bmatrix} [3,1] & [3] & [1] \\ [3] & [2] & [1] \\ [1] & [1] & [0] \end{bmatrix} = -\tfrac{1}{3}$$

$$= -\begin{bmatrix} [3,1] & [2,1] & [1] \\ [2,1] & [2] & [1] \\ [1] & [1] & [0] \end{bmatrix} \qquad (4.5)$$

$$= -\begin{bmatrix} [4,2] & [2,1] & [2,1] \\ [2,1] & [2] & [1] \\ [2,1] & [1] & [1^2] \end{bmatrix}$$

as in the matrix of Table III(c). It is also easy to verify the trace and determinant of the matrices of Table II and III from the character tables of S_N.

V. PLETHYSM AND PHASE CONVENTIONS

The operation of plethysm performs a symmetrization over all DCME related to each other by permutations of their rows and columns. One might wonder if these related DCME are equivalent up to a phase, and if so, what is the phase convention. In earlier work we had noted that the phase convention advanced by Baird and Biedenharn[5] for Gl(d) seemed plausible. That convention established a phase factor for each irrep

$$P_d(\lambda) \equiv \tfrac{1}{2}\sum_{i<j}^{d} ({}^i\lambda - {}^j\lambda), \text{ where } \lambda \equiv ({}^1\lambda, {}^2\lambda, \ldots, {}^d\lambda).$$

This reduces to the standard convention for SU(2), and the phase factor $P_d(\lambda) - P_d(\lambda_1) - P_d(\lambda_2)$ appropriate to a permutation in the order of coupling is independent of the dimension d and hence appropriate to coupling in the symmetric group. To be in accord with the plethysm operation one should require $P(\lambda) - 2P(\lambda_1)$ be even if $\lambda \in \lambda_1 \odot [2]$ and be odd if $\lambda \in \lambda_1 \odot [1^2]$. However some representations of the symmetric group are both even and odd, the simplest example of which is $[3,2,1] \in [2,1]\odot[2]$ and $[3,2,1] \in [2,1]\odot[1^2]$, whereas the above convention would indicate an even phase. The role played by the plethysm operation ($\alpha \odot \lambda$) in the symmetrization process, especially for $|\lambda| > 1$, indicates a simple phase convention is not entirely adequate. Even in cases in which the DCME can be evaluated using the orthogonality conditions the value is determined only up to a sign. Some relative phases are fixed by the orthogonality requirements but overall phases are not. Moreover from the point of view of $(n-j)$ coefficients in the general linear group one should equally consider permutations involving the first row and column of the DCME. It seems the entire question of phase can only be assessed when the development of coupling theory is more complete.

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Recoupling coefficients of the general linear group in bases adapted to shell theories

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Recoupling coefficients for tensor representations of the general linear group Gl(n) are identified with analogous quantities in representations of the symmetric group S_N. Two basis labeling schemes in Gl(n) are considered: (a) uses weights and outer product labels from S_N, and (b) uses outer plethysms in S_N and labels with respect to some elementary subgroup, usually SU(2). Scheme (a) corresponds to a generalized Gel'fand–Tsetlin basis and is the one usually adopted in elementary particle theories. Scheme (b) corresponds to the basis usually adopted in nuclear and atomic shell theory. The transformation between the two equivalent bases is identified with certain weighted double coset matrix elements (WDCME) of S_N. Racah factors are generalized isoscalar factors in scheme (a) and have previously been identified with certain WDCME in that basis. In scheme (b) Racah factors determine the coefficients of fractional parentage (CFP) and are here identified with certain double coset matrix elements (DCME) of S_N. Identification of these recoupling coefficients with the analogous quantities in S_N exposes new symmetries and orthogonality properties of the coefficients which follow from the representation theory of S_N. Some particular examples are verified by coefficients evaluated using well established techniques for SU(2).

I. INTRODUCTION

Racah recoupling coefficients of the unitary group U(n) are used extensively in elementary particle physics and in atomic and nuclear shell theory. Nuclear theorists for the most part have been the initiators of introducing results for the symmetric group S_N to simplify their shell calculations.[1] These applications have been mostly ad hoc and lack an explicit and/or general specification of the coupling in S_N. In our previous development[2] of the theory of coupling coefficients we have employed tensors of the general linear group Gl(n) explicitly labeled with respect to (w.r.t.) both S_N and Gl(n). The basis labeling scheme has utilized the generalized branching theorem which is equivalent to a generalized Yamanouchi[3] scheme in S_N and a generalized Gel'fand—Tsetlin[4] scheme in Gl(n). Although this is the scheme frequently used in elementary particle theories, it is not the preferred basis for spectroscopic shell theory. In this paper we bridge this gap by giving a more thorough analysis of the basis labeling schemes in Gl(n) and the relations between them. Two particular coefficients of importance to shell theory, Racah factors and coefficients of fractional parentage (CFP), are expressed in terms of double coset matrix elements (DCME) of S_N. A central result of our previous work and this paper is the identification of recoupling coefficients in Gl(n) with matrix elements in S_N. Aside from leading to the direct evaluation of certain of these coefficients, such identification allows the direct application of results that follow from the representation theory of S_N. Two such results are of prime importance: (1) Matrix representations of S_N can be assumed orthogonal and any transformation between equivalent representations is also orthogonal. Hence, the matrix elements are real. (2) The completeness and group orthogonality relations of S_N apply to the recoupling coefficients. In considering the recoupling process primitive tensors of Gl(n) simply provide a carrier space for the realization of irreps of S_N and the recoupling coefficients have significance independent of the dimension n.

Because the basic orthogonality we utilize derives from the theory of S_N, we will in this paper generally refer to (integral representations of) the noncompact group Gl(n) although the specific applications we have in mind are in the compact unitary and unitary unimodular subgroups U(n) and SU(n). The preceding paper gives a resume of the notation and pertinent results developed so far w.r.t. S_N. Section II analyzes the basis labeling problem in Gl(n). Section III considers the process of tensor coupling in Gl(n) and the utility of Racah's factorization lemma.[5] Coefficients of fractional parentage are related to Racah factors in Sec. IV. Section V identifies the transformation matrix between equivalent bases labeled as in elementary particle theory and as in shell theory with a weighted double coset matrix element (WDCME) in S_N. The Racah factor of shell theory is identified with a DCME in S_N.

II. BASIS LABELS OF Nth RANK TENSORS OF Gl (n)

The Schur—Weyl construction of Nth rank tensors of an n-dimensional defining vector space provides an irrep label λ, a partition of N into no more than n parts ($\lambda \vdash N$), that labels the irrep with respect to both S_N and Gl(n). The basis with respect to Gl(n) can be labeled by the weight $W=\{w_k\}$, $1 \le k \le n$, an Nth rank ordered n-tuple ($W \vdash N$), the components of which give the ranks of the totally symmetric subtensors in each of the n dimensions from which the weight is built up. The remaining basis labels w.r.t. Gl(n) can be indexed exactly as in the group S_N. I.e., let a primitive weight state be designated by the simple Nth rank product

$$|W\rangle \equiv \otimes |w_k\rangle, \tag{2.1}$$

which is by construction the basis of the identity irrep of $\otimes S_{w_k}$. All independent (and mutually orthogonal) states with the same weight W are realized by action of the coset representatives $S_N/\otimes S_{w_k}$ on the primitive weight state. By construction they are a basis for a representation of S_N, the representation induced by the identitity irrep of $\otimes S_{w_k}$. As shown in Sec. II of the pre-

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vious paper there is an isometry between inducing by action of the coset representatives and inducing by action of matrix basis projectors the second index of which is symmetry adapted to the subgroup $\otimes S_{w_k}$. The second index of the projector acts as an intermediate coupling specification of the weight and hence as a label w. r. t. $\mathrm{Gl}(n)$,

$$(\lambda \mid mm') |W\rangle \propto \left\langle \begin{matrix} & m & \\ N & \lambda & n \\ m' & (W) & \end{matrix} \right\rangle. \tag{2.2}$$

If one considers the basic defining space separated into two defining spaces of dimensions n_1 and $n_2 = n - n_1$ such that the primitive weight state factors as a product of two weights $|W\rangle = |W_1\rangle |W_2\rangle$ of ranks N_1 and N_2, then the labels in S_N are symmetry adapted to the subgroup $S_{N_1} \otimes S_{N_2}$. Thus in terms of the outer product in S_N the following labeling w. r. t. $\mathrm{Gl}(n)^N/\mathrm{Gl}(n_1)^{N_1} \otimes \mathrm{Gl}(n_2)^{N_2}$ corresponds to the general branching theorem

$$\left\langle \begin{matrix} & m & \\ N & \lambda & n_1 + n_2 \\ \lambda_1 & & \lambda_2 \\ m_1'(W_1) & & m_2'(W_2) \end{matrix} \right\rangle$$

$$\text{with } \lambda_1 \vdash N_1,\ \lambda_2 \vdash N_2,\ W = W_1 \dotplus W_2, \tag{2.3}$$

where W_1 is an N_1 rank n_1-tuple and W_2 is an N_2 rank n_2-tuple. The labeling scheme is a generalization of the Gel'fand–Tsetlin basis which starts with the dimension n and sequentially reduces it by one. In the same sense the analogous labeling scheme w. r. t. $S_N/S_{N_1} \otimes S_{N_2}$ is a generalization of the Yamanouchi basis. The third rank tensors of $\mathrm{Gl}(3)$ for example have $3^3 = 27$ independent elements which can be separated w. r. t. S_3 into a totally symmetric irrep [3] of dimension ten, a totally antisymmetric irrep $[1^3]$ of dimension one, and two [corresponding to the two values of m in Eq. (2.2)] eight-dimensional irreps [2, 1]. The weights correspond to 3-tuples (w_0, w_1, w_2) with $w_0 + w_1 + w_2 = 3$. The weight specification is sufficient to label the bases for the irreps [3] and $[1^3]$, and six of the basis members in the [2, 1] irrep with weights of the type (2, 1, 0). In the irreps [2, 1] two members of the basis have weights (1^3) and are differentiated by the two possible values of m' in Eq. (2.2).

Similarly the inner product in S_N provides a labeling w. r. t. $\mathrm{Gl}(n_1 n_2)^N/\mathrm{Gl}(n_1)^N \otimes \mathrm{Gl}(n_2)^N$ as

$$\left\langle \begin{matrix} & m & \\ N & \lambda & n_1 n_2 \\ \mu & & \nu \\ m_1'(W_1) & & m_2'(W_2) \end{matrix} \right\rangle \tag{2.4}$$

with $\mu \vdash N$, $(\lambda)\epsilon(\mu)\mathrm{x}(\nu)$, $\nu \vdash N$, $W = W_1 W_2$, where W_1 is an N rank n_1-tuple and W_2 is an N rank n_2-tuple. The present paper does not exploit this labeling scheme.

In shell theory applications one forms symmetrized species out of a basic defining representation that can itself be considered as a symmetrized tensor of a more fundamental defining space. For example in atomic shell theory one has the following decompositions for equivalent p or d electrons:

$$(p)^2 \to {}^1D, {}^1S, {}^3P, \quad (d)^2 \to {}^1G, {}^1D, {}^1S, {}^3F, {}^3P,$$

$$(p)^3 \to {}^2D, {}^2P, {}^4S, \quad (d)^3 \to {}^2H, {}^2G, {}^2F, 2{}^2D, {}^2P, {}^4F, {}^4P.$$

The symmetrization of a tensor basis is termed an outer plethysm by Littlewood due to its action as a symmetrized outer product in the symmetric group. Corresponding to the above third rank tensors one has the plethysms:

$$[2] \odot [3] = [6] + [4, 2] + [2^3],$$

$$[2] \odot [2, 1] = [5, 1] + [4, 2] + [3, 2, 1],$$

$$[2] \odot [1^3] = [3^2] + [4, 1^2],$$

$$[4] \odot [3] = [12] + [10, 2] + [9, 3] + [8, 4] + [6^2],$$

$$[4] \odot [2, 1] = [11, 1] + [10, 2] + [9, 3] + 2[8, 4] + [7, 5],$$

$$[4] \odot [1^3] = [9, 3] + [7, 5],$$

where in the decomposition of the latter three plethysms we have listed only the bipartition irreps. Nuclear and atomic shell theories decompose plethysms w. r. t. SU(2), i. e., the subgroup sequence involved is $S_{2jN} \circledS \mathrm{SU}(2)^{2jN}/S_N \circledS [S_{2j} \circledS \mathrm{SU}(2)^{2j}]^N$. Symmetrization within a defining SU(2) space requires the plethysm be decomposed only up to bipartition representations. Furthermore because of the Pauli principle the symmetrizing irreps can have parts no larger than four for nuclear isospin shell theory, and two for atomic shell theory. Thus $[(2)] \odot [(3)]$ and $[(4)] \odot [(3)]$ have no physical realization in atomic shell theory, but they do occur in nuclear shell theory. The bipartitions contained in the above plethysms [2, 1] and $[1^3]$ correspond to the doublet and quartet decompostions in atomic shell theory.

Shell theory requires the decomposition of plethysms of totally symmetric irreps $[2j]$ of SU(2). The corresponding tensor labels are

$$\left\langle \begin{matrix} & m & \\ N & \lambda & 2j+1 \\ & J & \\ & M & \end{matrix} \right\rangle$$

where the angular momentum J indicates the bipartition $[jN + J, jN - J]$ with basis $[jN + M] \otimes [jN - M]$, $-J \leq M \leq J$. In the general case J and M may be considered as simply labels w. r. t. any defining subgroup $\mathrm{SU}(n')$. One of the principal results of this paper is to identify the unitary transformation between the basis sets

$$\left\langle \begin{matrix} & m & \\ N & \lambda & 2j+1 \\ & m'(W) & \end{matrix} \right\rangle \text{ and } \left\langle \begin{matrix} & m & \\ N & \lambda & 2j+1 \\ & J & \\ & M & \end{matrix} \right\rangle$$

with a WDCME in the symmetric group. The defining irrep of $\mathrm{Gl}(2j+1)$ is the identity (in S_N) tensor irrep $[2j]$ of SU(2). The $2j+1$ dimensions expressed in terms of the two-dimensional weights (α, β) of SU(2) are (α^{2j-k}, β^k) with $0 \leq k \leq 2j$. The weights of the Nth rank tensor basis in $\mathrm{Gl}(2j+1)$ are given by $2j+1$ tuples $W = \{w_k\}$ such that $\sum w_k = N$ and $\sum k w_k = jN - M$. The desired unitary transformation therefore has rows and columns designated by $m'(W)$ and J with λ, M, and j acting as parameters. For example, the 6×6 matrix for $j=1$, $\lambda=[2]$ is unit diagonal for $M\neq 0$ and 2×2 for $M=0$ with $W=(0,2,0)$, $(1,0,1)$, and $J=2,0$. The 10×10 matrix for $j=1$, $\lambda=[3]$ is 2×2 in the blocks $M\pm 1$, $W=(2,0,1)$, $(1,2,0)$; and $M=0$, $W=(0,3,0)$, (1^3) with $J=3,1$. The 8×8 matrix for $j=1$, $\lambda=[2,1]$ blocks into two 1×1 for $M=\pm 2$, two 2×2 for $M=\pm 1$, and a 2×2 for $M=0$ that further decomposes for reasons discussed in the final section. The nontrivial matrices are displayed in Table I of Sec. V. Note that additional multiplicity labels may be necessary to completely specify the basis in either scheme. Such is the case for the irrep $[8,4]$ in the decomposition of $[4]\odot[2,1]$ for which a seniority label[5] can be used to differentiate the multiplicity. The problems of multiplicity labels is further discussed in the Appendix to Ref. 2(a). However the multiplicity is resolved, such labels can simply be appended to the symbols discussed here.

III. TENSOR COUPLING IN S_N AND Gl(n)

Tensor coupling can be accomplished by using the Clebsch—Gordan coefficients of the group and/or using matrix basis projectors in S_N. The result of projecting on a tensor product is the tensor sum

$$(\lambda\,|k,\lambda_j m_j)\otimes\left\langle\begin{matrix} & m_j & \\ N_j & \lambda_j & n \\ & M_j & \end{matrix}\right\rangle=\sum_M\left\langle\begin{matrix} & k & \\ N & \lambda & n \\ & M & \end{matrix}\right\rangle\left(\begin{matrix}\lambda \\ M\end{matrix}\,\middle|\,\begin{matrix}\lambda_j \\ M_j\end{matrix}\right)_n , \quad (3.1)$$

where

$$\left(\begin{matrix}\lambda \\ M\end{matrix}\,\middle|\,\begin{matrix}\lambda_j \\ M_j\end{matrix}\right)_n$$

is the Clebsch—Gordan coefficient in SU(n). Equation (3.1) is readily verified by multiplying both sides by

$$\left(\begin{matrix}\lambda_j \\ M_j\end{matrix}\,\middle|\,\begin{matrix}\lambda' \\ M'\end{matrix}\right)_n$$

and summing on M_j to obtain the identity

$$(\lambda\,|k,\lambda_j m_j)\otimes\sum\left\langle\begin{matrix} & m_j & \\ N_j & \lambda_j & n \\ & M_j & \end{matrix}\right\rangle\left(\begin{matrix}\lambda_j \\ M_j\end{matrix}\,\middle|\,\begin{matrix}\lambda' \\ M'\end{matrix}\right)_n$$

$$=(\lambda\,|k,\lambda_j m_j)\left\langle\begin{matrix} & \lambda_j m_j & \\ N & \lambda' & n \\ & M' & \end{matrix}\right\rangle$$

$$=\delta^{\lambda\lambda'}\left\langle\begin{matrix} & k & \\ N & \lambda & n \\ & M' & \end{matrix}\right\rangle. \quad (3.2)$$

Whenever the basis label M has explicit subgroup significance, a partial Clebsch—Gordan coupling can be carried out w.r.t. this subgroup. This leads to a direct (Racah) factorization of the coupling coefficients which we proceed to demonstrate for the two labeling schemes considered here. For a basis labeled w.r.t. the general branching scheme we may define an isoscalar factor as

$$\sum_{{}_iM_j}\left\langle\begin{matrix} & m_1 & \\ N_1 & \lambda_1 & {}_1n+{}_2n \\ & {}_1\lambda_1\quad{}_2\lambda_1 & \\ & {}_1M_1\quad{}_2M_1 & \end{matrix}\right\rangle\left\langle\begin{matrix} & m_2 & \\ N_2 & \lambda_2 & {}_1n+{}_2n \\ & {}_1\lambda_2\quad{}_2\lambda_2 & \\ & {}_1M_2\quad{}_2M_2 & \end{matrix}\right\rangle$$

$$\times\left(\begin{matrix}{}_1\lambda_1 & {}_1\lambda_2 \\ {}_1M_1 & {}_1M_2\end{matrix}\,\middle|\,\begin{matrix}{}_1\lambda \\ {}_1M\end{matrix}\right)_{{}_1n}\left(\begin{matrix}{}_2\lambda_1 & {}_2\lambda_2 \\ {}_2M_1 & {}_2M_2\end{matrix}\,\middle|\,\begin{matrix}{}_2\lambda \\ {}_2M\end{matrix}\right)_{{}_2n}$$

$$\equiv\sum_\lambda\left\langle\begin{matrix} & m_1\quad m_2 & \\ & \lambda_1\quad\lambda_2 & \\ N & \lambda & {}_1n+{}_2n \\ & {}_1\lambda\quad{}_2\lambda & \\ & {}_1M\quad{}_2M & \end{matrix}\right\rangle$$

$$\times\left[\begin{matrix}\lambda \\ {}_1\lambda\quad{}_2\lambda\end{matrix}\,\middle|\,\begin{matrix}\lambda_1 & & \lambda_2 \\ {}_1\lambda_1\quad{}_2\lambda_1 & & {}_1\lambda_2\quad{}_2\lambda_2\end{matrix}\right]. \quad (3.3)$$

The isoscalar factor does not depend on the subgroup basis indices ${}_iM$ as can be ascertained by considering the action of raising or lowering operators of SU(${}_1n$) $\otimes$ SU(${}_2n$) on both sides of Eq. (3.3). Projecting in S_N and bringing the coupling coefficients of the subgroup to the right side via the unitarity conditions gives the usual factored form

$$\left(\begin{matrix}\lambda \\ {}_1\lambda\quad{}_2\lambda \\ {}_1M\quad{}_2M\end{matrix}\,\middle|\,\begin{matrix}\lambda_1 & & \lambda_2 \\ {}_1\lambda_1\quad{}_2\lambda_1 & & {}_1\lambda_2\quad{}_2\lambda_2 \\ {}_1M_1\quad{}_2M_1 & & {}_1M_2\quad{}_2M_2\end{matrix}\right)_{{}_1n+{}_2n}$$

$$=\left[\begin{matrix}\lambda \\ {}_1\lambda\quad{}_2\lambda\end{matrix}\,\middle|\,\begin{matrix}\lambda_1 & & \lambda_2 \\ {}_1\lambda_1\quad{}_2\lambda_1 & & {}_1\lambda_2\quad{}_2\lambda_2\end{matrix}\right]$$

$$\times\left(\begin{matrix}{}_1\lambda \\ {}_1M\end{matrix}\,\middle|\,\begin{matrix}{}_1\lambda_1 & {}_1\lambda_2 \\ {}_1M_1 & {}_1M_2\end{matrix}\right)_{{}_1n}\left(\begin{matrix}{}_2\lambda \\ {}_2M\end{matrix}\,\middle|\,\begin{matrix}{}_2\lambda_1 & {}_2\lambda_2 \\ {}_2M_1 & {}_2M_2\end{matrix}\right)_{{}_2n}. \quad (3.4)$$

The isoscalar factor is unitary on the labels $(\lambda, {}_i\lambda_j)$ and has been identified[2(d)] with the WDCME on the symmetric group as

$$\left[\begin{matrix}\lambda \\ {}_i\lambda\end{matrix}\,\middle|\,\begin{matrix}\lambda_j \\ {}_i\lambda_j\end{matrix}\right]=\left(\frac{|\lambda|\,{}_iN!N_j!\;|{}_i\lambda_j|}{N!\;|{}_i\lambda|\;|\lambda_j|\,{}_iN_j!}\right)^{1/2}\left[\begin{matrix}\lambda & \lambda_j \\ {}_i\lambda & {}_i\lambda_j\end{matrix}\right] \quad (3.5)$$

showing the isoscalar factor may be considered real and independent of the dimension n.

In the basis using the plethysm operation similar arguments lead to the factorization

$$\left(\begin{matrix}\lambda \\ J \\ M\end{matrix}\,\middle|\,\begin{matrix}\lambda_1 & \lambda_2 \\ J_1 & J_2 \\ M_1 & M_2\end{matrix}\right)_n=\left[\begin{matrix}\lambda \\ J\end{matrix}\,\middle|\,\begin{matrix}\lambda_1 & \lambda_2 \\ J_1 & J_2\end{matrix}\right]_n\left(\begin{matrix}J \\ M\end{matrix}\,\middle|\,\begin{matrix}J_1 & J_2 \\ M_1 & M_2\end{matrix}\right)_{n'}. \quad (3.6)$$

Here again although symbols J and M normally associated with SU(2) are used they may be considered generically w.r.t. to any fundamental defining group SU(n'). Since it is the transformation between equivalent orthogonal bases, the square bracket is unitary on the labels (λ, J_j). A second principal result of this paper is to connect the Racah factor

$$\begin{bmatrix} \lambda & \lambda_1 & \lambda \\ J & J_1 & J \end{bmatrix}_n$$

to a DCME in the symmetric group. Some examples are given in Table II of Sec. V. We first demonstrate that these Racah factors determine the coefficients of fractional parentage of shell theory.

IV. COEFFICIENTS OF FRACTIONAL PARENTAGE (CFP)

In the LS coupling scheme the CFP factor into two parts each being a modified Clebsch–Gordan coefficient in the unitary group of dimension appropriate to the external (orbital) or the internal (spin-isospin) space. Using the result of Sec. III these modified Clebsch–Gordan coefficients are identified as the Racah factors for this labeling scheme.

For Fermi particles the external and internal spaces are coupled to form the totally antisymmetric irrep using the Clebsch–Gordan coefficient in S_N

$$\left(\begin{array}{cc|c} \lambda & \lambda' & [1^N] \\ m & m' & 0 \end{array}\right)_{S_N} = \frac{\Lambda^{\lambda}_{m}\delta^{\lambda'\tilde{\lambda}}\delta_{m'\tilde{m}}}{|\lambda|^{1/2}}, \tag{4.1}$$

where Λ^{λ}_{m} is a phase such that for subgroup specification $m \to {}_1\lambda\, {}_1m,\ {}_2\lambda\, {}_2m$ the product phase

$$\Lambda^{\lambda}_{{}_1\lambda_1 m_2\lambda_2 m}\,\Lambda^{{}_1\lambda}_{{}_1m}\,\Lambda^{{}_2\lambda}_{{}_2m} = \phi(\lambda, {}_1\lambda, {}_2\lambda) \tag{4.2}$$

is independent of ${}_1m, {}_2m$. This is the only significant effect of phase on what follows.

The CFP is defined as the scalar product of an antisymmetrized state

$|N\lambda\alpha\beta LM_L SM_S\rangle_a$

$$\equiv \sum_{\substack{{}_1\lambda_2\lambda \\ {}_1m_2m}} \frac{\Lambda^{\lambda}_{{}_1\lambda_1 m,\,{}_2\lambda_2 m}}{|\lambda|^{1/2}} \times \left\langle \begin{array}{ccc} & {}_1m \quad {}_2m & \\ & {}_1\lambda \quad {}_2\lambda & \\ N & \lambda & n \\ & \alpha \quad L & \\ & M_L & \end{array} \right\rangle \left\langle \begin{array}{ccc} & {}_1\tilde{m} \quad {}_2\tilde{m} & \\ & {}_1\tilde{\lambda} \quad {}_2\tilde{\lambda} & \\ N & \tilde{\lambda} & n' \\ & \beta \quad S & \\ & M_S & \end{array} \right\rangle \tag{4.3}$$

and a partially symmetrized state

$|N({}_1\lambda_2\lambda)({}_1\alpha_2\alpha)({}_1\beta_2\beta)({}_1L_2L)LM_L({}_1S_2S)SM_S\rangle$

$$\equiv \sum_{\substack{{}_1m_2m \\ {}_1M_L\, {}_2M_L \\ {}_1M_S\, {}_2M_S}} \frac{\Lambda^{{}_1\lambda}_{{}_1m}\Lambda^{{}_2\lambda}_{{}_2m}}{(|{}_1\lambda|\,|{}_2\lambda|)^{1/2}} \times \left\langle \begin{array}{ccc} & {}_1m & \\ {}_1N_1 & \lambda & n \\ & {}_1\alpha \quad {}_1L & \end{array} \right\rangle \left\langle \begin{array}{ccc} & {}_1m & \\ {}_1N & {}_1\tilde{\lambda} & n' \\ & {}_1\beta \quad {}_1S & \end{array} \right\rangle$$

$$\times \left\langle \begin{array}{ccc} & {}_2m & \\ {}_2N & {}_2\lambda & n \\ & {}_2\alpha \quad {}_2L & \end{array} \right\rangle \left\langle \begin{array}{ccc} & {}_2\tilde{m} & \\ {}_2N & {}_2\tilde{\lambda} & n' \\ & {}_2\beta \quad {}_2S & \end{array} \right\rangle \times \left(\begin{array}{cc|c} {}_1L & {}_2L & L \\ {}_1M_L & {}_2M_L & M_L \end{array}\right)_2 \left(\begin{array}{cc|c} {}_1S & {}_2S & S \\ {}_1M_S & {}_2M_S & M_S \end{array}\right)_2. \tag{4.4}$$

In Eq. (4.4) multiplicity labels α, β are specifically exhibited. Expanding the Nth rank tensors of Eq. (4.3) by Clebsch–Gordan coupling in $U(n)$ as

$$\left\langle \begin{array}{ccc} & {}_1m \quad {}_2m & \\ & {}_1\lambda \quad {}_2\lambda & \\ N & \lambda & n \\ & \alpha L & \\ & M_L & \end{array} \right\rangle = \sum_{\substack{{}_1\alpha_2\alpha \\ {}_1L_2L \\ {}_1M_L\, {}_2M_L}} \left\langle \begin{array}{ccc} & {}_1m & \\ {}_1N & {}_1\lambda & n \\ & {}_1\alpha \quad {}_1L & \\ & {}_1M_L & \end{array} \right\rangle \left\langle \begin{array}{ccc} & {}_2m & \\ {}_2N & {}_2\lambda & n \\ & {}_2\alpha \quad {}_2L & \\ & {}_2M_L & \end{array} \right\rangle \times \left(\begin{array}{cc|c} {}_1\lambda & {}_2\lambda & \lambda \\ {}_1\alpha\ {}_1L & {}_2\alpha\ {}_2L & \alpha L \\ {}_1M_L & {}_2M_L & M_L \end{array}\right)_n$$

and taking the scalar product of Eq. (4.4) on Eq. (4.3) gives

$$\langle N({}_1\lambda_2\lambda)({}_1\alpha_2\alpha)({}_1\beta_2\beta)({}_1L_2L)L'M'_L({}_1S_2S) \times S'M'_S|N\lambda\alpha\beta LM_L SM_S\rangle_a = \delta_{L'L}\delta_{M'_L M_L},$$

$$\langle({}_1\lambda_2\lambda)({}_1\alpha_2\alpha)({}_1\beta_2\beta)({}_1L_2L)({}_1S_2S)|\}\lambda\alpha\beta LS\rangle$$

$$= \sum_{\substack{{}_1M_L\, {}_2M_L \\ {}_1M_S\, {}_2M_S}} \phi(\lambda_1\lambda_2\lambda)\left(\frac{|{}_1\lambda|\,|{}_2\lambda|}{|\lambda|}\right)^{1/2} \times \left(\begin{array}{c|cc} L' & {}_1L & {}_2L \\ M'_L & {}_1M_L & {}_2M_L \end{array}\right)_2 \left(\begin{array}{c|cc} S' & {}_1S & {}_2S \\ M'_S & {}_1M_S & {}_2M_S \end{array}\right)_2$$

$$\times \left(\begin{array}{cc|c} {}_1\lambda & {}_2\lambda & \lambda \\ {}_1\alpha\ {}_1L & {}_2\alpha\ {}_2L & \alpha\ L \\ {}_1M_L & {}_2M_L & M_L \end{array}\right)_n \times \left(\begin{array}{cc|c} {}_1\tilde{\lambda} & {}_2\tilde{\lambda} & \tilde{\lambda} \\ {}_1\beta_1 S & {}_2\beta_2 S & \beta\ S \\ {}_1M_S & {}_2M_S & M_S \end{array}\right)_{n'}. \tag{4.5}$$

Use of Eq. (3.6) allows the CFP to be expressed as

$\langle({}_1\lambda_2\lambda)({}_1\alpha_2\alpha)({}_1\beta_2\beta)({}_1L_2L)({}_1S_2S)|\}\lambda\alpha\beta LS\rangle$

$$= \left(\frac{|_1\lambda|\,|_2\lambda|}{|\lambda|}\right)^{1/2} \phi(\lambda_1\lambda_2\lambda) \begin{bmatrix} {}_1\lambda & {}_2\lambda & \lambda \\ {}_1\alpha_1 L & {}_2\alpha_2 L & \alpha l \end{bmatrix}_n \times \begin{bmatrix} {}_1\tilde{\lambda} & {}_2\tilde{\lambda} & \tilde{\lambda} \\ {}_1\beta\ {}_1S & {}_2\beta\ {}_2S & \beta\ S \end{bmatrix}_{n'}. \quad (4.6)$$

For the atomic case $n'=2$, $n=2l+1$, the last factor is trivial and writing ${}_1\lambda = {}_1\tilde{S}$, etc., we have

$$\langle({}_1\alpha_2\alpha)({}_1L_2L)({}_1S_2S)\,|\}\,\alpha LS\rangle_l$$

$$= \pm\left\{\left[\left(\frac{N}{2}+S+1\right)!\left(\frac{N}{2}-S\right)!\ {}_1N!\ {}_2N!\ (2_1S+1)(2_2S+1)\right] \Big/ \left(\frac{{}_1N}{2}+{}_1S+1\right)!\left(\frac{{}_1N}{2}-{}_1S\right)!\left(\frac{{}_2N}{2}+{}_2S+1\right)! \times\left(\frac{{}_2N}{2}-{}_2S\right)!\ (2S+1)\right]\right\}^{1/2} \times \begin{bmatrix} {}_1\tilde{S} & {}_2\tilde{S} & \tilde{S} \\ {}_1\alpha_1 L & {}_2\alpha_2 L & \alpha l \end{bmatrix}_l. \quad (4.7)$$

CFP are usually defined with ${}_2N$ restricted to unit value.

V. CONNECTION WITH DC OF THE SYMMETRIC GROUP

A. Basis transformation and WDCME

The unitary transformation between the basis utilizing the general branching law and that using the operation of plethysm involves the double coset reduction

$$[S_{1^n} \circledS \mathrm{Gl}(d_1)^{1^n}] \otimes [S_{2^n} \circledS \mathrm{Gl}(d_2)^{2^n}] \backslash S_{nN} \circledS \mathrm{Gl}(d)^{nN} / S_N \circledS [S_n \circledS \mathrm{Gl}(d)^n]^N. \quad (5.1)$$

For the purposes of shell theory we may take $d=2$, $d_1 = 1 = d_2$, and $n=2j$. Since the general linear group simply provides a carrier space for the irreps of the symmetric group we may restrict our considerations to the latter group. The DC symbol is the type Eq. (4.1) of the previous paper

$$\begin{bmatrix} S_{nN} & S_N \circledS (S_n)^N \\ S_{1^n} & \\ S_{2^n} & \otimes (S_{w_k} \circledS [S_{n-k} \otimes S_k]^{w_k}) \end{bmatrix} \quad (5.2)$$

and is designated by the Nth rank $n+1$-tuple $\{w_k\}$, $0 \le k \le n$, with $\sum w_k = N$ and $\sum_k kw_k = {}_2n$. For shell theory applications the DCME are of the form

$$\begin{bmatrix} [jN+J,\ jN-J] & [2j] \odot [\lambda] \\ [jN+M] & \otimes([2j-k] \odot [w_k]) \\ [jN-M] & \otimes([k] \odot [w_k]) \end{bmatrix}, \quad (5.3)$$

where the dimensionality of the underlying linear groups limit the irreps to the totally symmetric single partitions, or the bipartitions as indicated. The equivalent bases are those generated by Mackey's subgroup theorem. The WDCME has been shown to be the unitary transformation between these bases. The matrix elements have the form

$$\left(\frac{(2J+1)(jN+M)!\,(jN-M)!\,N!\,(2j!)^N}{(jN+J+1)!\ (jN-J)!\ |\lambda|\otimes[w_k!\,[(2j-k)!\ k!]^{w_k}]}\right)^{1/2} \times \begin{bmatrix} [jN+J,\ jN-J] & [2j] & \odot[\lambda] \\ [jN+M] & \otimes([2j-k] & \odot[w_k]) \\ [jN-M] & \otimes([k] & \odot[w_k]) \end{bmatrix}, \quad (5.4)$$

where j, M, and λ are parameters and the rows and columns are designated by $(J, \{w_k\})$. The matrix elements for $J=jN$ evaluate directly as the weighting factor. The orthogonality relations can be used to evaluate any two by two matrix that includes $J=jN$. As examples, Table I gives some of the transformations between the equivalent bases discussed in Sec. II.

It is to be noted that only the relative phase is determined by orthogonality and in Table I we have chosen a convention with positive determinant. Besides the totally symmetric irrep $J=jN$, the DCME is also trivial for $N=2$, $w_0=1=w_{2j}$ (it is the matrix element of the identity operator) leading to the direct evaluation of the WDCME in those cases. For $M=0$ one may use the DC decomposition $S_2 \circledS (S_{jN})^2 \backslash S_{2jN}/S_N \circledS (S_{2j})^N$. This leads to the additional diagonalization indicated for $j=1$, $\lambda=[2,1]$; $j=2$, $\lambda=[3]$; and $j=2$, $\lambda=[2,1]$. In these cases an additional factor of two may appear in the radical of Eq. (5.4) if $w_k \ne w_{2j-k}$ such as $j=2$, $M=0$, $\{w_k\} = (0,2,0,0,1)$. A subscript s or a is addended to the weight to indicate a symmetric or antisymmetric combination. Such a subscript is also required for $M=0$, $\{w_k\} = (1^3)$, to distinguish the basis members in the two-dimensional $\lambda=[2,1]$ irrep. The dash simply indicates the matrix element is not easily calculable by the present considerations. The matrix for $j=2$, $\lambda=[2,1]$, $M=0$ is presented to demonstrate the diagonalization and to emphasize that additional multiplicity labeling, such as a seniority label to distinguish $J=2$, may be required to uniquely specify the transformation.

B. Racah factor and DCME

The Racah factor for the shell theory labeling scheme corresponds to the DC decomposition

$$[S_{2jN_1} \circledS \mathrm{Gl}(2)^{2jN_1}] \circledS [S_{2jN_2} \circledS \mathrm{Gl}(2)^{2jN_2}] \backslash S_{2jN} \circledS \mathrm{Gl}(2)^{2jN} / S_N \circledS [S_{2j} \circledS \mathrm{Gl}(2)^{2j}]^N,$$

for that DC with intertwining subgroup

$$[S_{N_1} \circledS [S_{2j} \circledS \mathrm{Gl}(2)^{2j}]^{N_1}] \otimes [S_{N_2} \circledS [S_{2j} \circledS \mathrm{Gl}(2)^{2j}]^{N_2}].$$

The DC symbol is of the form of Eq. (5.2) with $w_0 = N_1$, $w_{2j} = N_2$. The DCME has the form

$$\begin{bmatrix} [jN+J,\quad jN-J] & [2j] \odot (\lambda) \\ [jN_1+J_1,\ jN_1-J_1] & [2j] \odot (\lambda_1) \\ [jN_2+J_2,\ jN_2-J_2] & [2j] \odot (\lambda_2) \end{bmatrix} = \begin{bmatrix} \lambda & \lambda_1 & \lambda_2 \\ J & J_1 & J_2 \end{bmatrix}_j. \quad (5.5)$$

For fixed j, J, λ_1, and λ_2 the DCME is orthogonal with rows and columns indicated by $(J_1 J_2, \lambda)$. This being the unitary transformation between the equivalent basis schemes, the DCME is identified with the Racah factor. Whenever for fixed J, j, λ_1, and λ_2 the intertwining is unique, the DCME is trivially ± 1. Such is always the case for the DCME with $J=jN$ or $J=jN-1$. Examination of the plethysms for $N=3$ evaluated in Sec. II

TABLE I. Transformation matrices: WDCME of Eq. (5.4); j, λ, M parameters, rows and columns designated by (J, w_k).

$j=1,\ \lambda=[2],\ M=0$

J \ w_k	(0, 2, 0)	(1, 0, 1)
2	$(2/3)^{1/2}$	$(1/3)^{1/2}$
0	$-(1/3)^{1/2}$	$+(2/3)^{1/2}$

$j=1,\ \lambda=[3],\ M=1$

J \ w_k	(2, 0, 1)	(1, 2, 0)
3	$(1/5)^{1/2}$	$(4/5)^{1/2}$
1	$-(4/5)^{1/2}$	$+(1/5)^{1/2}$

$j=1,\ \lambda=[3],\ M=0$

J \ w_k	(0, 3, 0)	(1, 1, 1)
3	$(2/5)^{1/2}$	$(3/5)^{1/2}$
1	$-(3/5)^{1/2}$	$(2/5)^{1/2}$

$j=1,\ \lambda=[2,1],\ M=1$

J \ w_k	(2, 0, 1)	(1, 2, 0)
2	$1/\sqrt{2}$	$1/\sqrt{2}$
1	$-1/\sqrt{2}$	$1/\sqrt{2}$

$j=1,\ \lambda=[2,1],\ M=0$

J \ w_k	$(1, 1, 1)_s$	$(1, 1, 1)_{1a}$
2	1	0
1	0	1

$j=2,\ \lambda=[2],\ M=2$

J \ w_k	(1, 0, 1, 0, 0)	(0, 2, 0, 0, 0)
4	$(3/7)^{1/2}$	$(4/7)^{1/2}$
2	$-(4/7)^{1/2}$	$(3/7)^{1/2}$

$j=2,\ \lambda=[2],\ M=1$

J \ w_k	(1, 0, 0, 1, 0)	(0, 1, 1, 0, 0)
4	$(1/7)^{1/2}$	$(6/7)^{1/2}$
2	$-(6/7)^{1/2}$	$(1/7)^{1/2}$

$j=2,\ \lambda=[2],\ M=0$

J \ w_k	(1, 0, 0, 0, 1)	(0, 1, 0, 1, 0)	(0, 0, 2, 0, 0)
4	$(1/35)^{1/2}$	$(16/35)^{1/2}$	$(18/35)^{1/2}$
2	$(4/7)^{1/2}$	$(1/7)^{1/2}$	$-(2/7)^{1/2}$
0	$-(2/5)^{1/2}$	$(2/5)^{1/2}$	$-(1/5)^{1/2}$

$j=2,\ \lambda=[1^2],\ M=0$

J \ w_k	(1, 0, 0, 0, 1)	(0, 1, 0, 1, 0)
3	$(1/5)^{1/2}$	$(4/5)^{1/2}$
1	$-(4/5)^{1/2}$	$(1/5)^{1/2}$

$j=2,\ \lambda=[3],\ M=0$

J \ w_k	(1, 0, 1, 0, 1)	(0, 1, 1, 1, 0)	(0, 0, 3, 0, 0)	(1, 0, 0, 2, 0) $(0, 2, 0, 0, 1)_s$	(0, 2, 0, 0, 1) $(1, 0, 0, 2, 0)_a$
6	$(3/77)^{1/2}$	$(48/77)^{1/2}$	$(18/77)^{1/2}$	$(8/77)^{1/2}$	0
4	—	—	—	—	0
2	—	—	—	—	0
0	—	—	—	—	0
3	0	0	0	0	1

$j=2,\ \lambda=[2,1],\ M=0$

J \ w_k	(1, 0, 0, 2, 0) $(0, 2, 0, 0, 1)_s$	$(1, 0, 1, 0, 1)_s$	$(0, 1, 1, 1, 0)_s$	(1, 0, 0, 2, 0) $(0, 2, 0, 0, 1)_a$	$(1, 0, 1, 0, 1)_a$	$(0, 1, 1, 1, 0)_a$
4	—	—	—	0	0	0
2	—	—	—	0	0	0
2	—	—	—	0	0	0
5	0	0	0	—	—	—
3	0	0	0	—	—	—
1	0	0	0	—	—	—

— means the entry is not easily evaluated by the techniques described in this paper.

shows all DCME are trivial except for those listed in Table II. The DC matrices listed in Table II are two by two except for a three by three matrix for $j=2$, $J=2$, $\lambda_1=[2]$. Although the DCME are not directly calculable by methods discussed in this paper, the orthogonality of the WDCME does determine the values for $j=1$, $J=1$, $\lambda_1=[2]$ and this matrix is given in Table II(a). The double occurrence of the label $[2,1]$ in the DCME for irrep $[8,4]$ indicates the necessity of a seniority multiplicity label to uniquely differentiate the DCME in that case.

For the case $N_1=N_2$ the DC decomposition $S_2 \circledS (S_{2jN_1})^2 \backslash S_{4jN_1}/S_{2N_1} \circledS (S_{2j})^{2N_1}$ with intertwining subgroup $S_2 \circledS [S_{N_1} \circledS (S_{2j})^{N_1}]^2$ may be used. This allows $N_1=1=N_2$ to be considered trivially and introduces some simplification in cases of higher rank.

The WDCME and the DCME of this section both reduce to the unit factor for $j=\frac{1}{2}$ as should be the case. Interestingly the WDCME has been identified as forming the transformation matrix between equivalent bases labeled by weights and the generalized branching theorem on the one hand and by a plethysm scheme on the other. The DCME forms the Racah factor in the plethysm scheme. Our earlier work has shown the DCME forms the transformation matrix between equiva-

TABLE II. Row and column labels of Racah factors DCME of Eq. (5.5) for $N=3$, $[2j]=[2]$ or $[4]$ $\left[\begin{smallmatrix}\lambda \\ J\end{smallmatrix} \middle| \begin{smallmatrix}\lambda_1 & \lambda_2 \\ J_1 & J_2\end{smallmatrix}\right]_j$.

(a) $[2j]=[2]$ $J=1\sim[4,2]$ $\lambda_1=[2]$ $\lambda_2=[1]$ $\therefore$ $J_2=1$

J_1 \ λ	[3]	[2, 1]
2	$(4/9)^{1/2}$	$(5/9)^{1/2}$
0	$-(5/9)^{1/2}$	$(4/9)^{1/2}$

(b) $[2j]=[4]$ $\lambda_2=[1]$ $\therefore$ $J_2=2$

J		$\lambda_1=[2]$	$\lambda_1=[1^2]$
$4\sim[10,2]$	λ	[3], [2, 1]	
	J_1	4, 2	
$3\sim[9,3]$	λ	[3], [2, 1]	[2, 1], $[1^3]$
	J_1	4, 2	3,1
$2\sim[8,4]$	λ	[3], [2, 1], [2, 1]	[2, 1], [2, 1]
	J_1	4, 2, 0	3, 1
$1\sim[7,5]$	λ		[2, 1], $[1^3]$
	J_1		3, 1

lent bases labeled by different generalized branching schemes and the WDCME forms the isoscalar (Racah) factor in these schemes. Two outstanding problems in our general development remain: exposition of a consistent phase scheme in $\mathrm{Gl}(n)$ and S_N exhibiting to what extent relative phases are fixed and which phase conventions are arbitrary, and elucidation of an algorithm for the convenient evaluation of a general DCME in S_N.

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Unitarily equivalent multiparticle Hamiltonian systems yielding equal scattering for corresponding states

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Consider a pair of nonrelativistic N-particle ($N \geq 2$) systems with unitarily equivalent Hamiltonians H and W^*HW. Typically, W^*HW involves nonlocal multiparticle interactions even when only local pair interactions are present in H. For the respective cases when H involves short-range interactions alone and also when it involves long-range ones, sufficient conditions are established in order for a scattering amplitude of the first system pertaining to any states in given entry and exit channels to equal the amplitude of the second system pertaining to corresponding states. This is accomplished by a time-dependent approach. The correspondence in question assigns to each channel state of the first system a channel state of the second system in a bijective and intuitively natural manner. Nontrivial examples are given of unitary operators W for which the above equality holds for all channels of these systems. This applies to a large class of interactions in H, including interactions with suitable long-range parts. The present work is the theoretical foundation of a new method of the authors, discussed elsewhere, for investigating many-body nuclear forces phenomenologically.

1. INTRODUCTION

In a previous publication,[1] we derived necessary and sufficient conditions for two unitarily equivalent nonrelativistic Hamiltonians to yield the same S operator for single-channel scattering. Our work was based on a time-dependent approach similar to that of Ekstein,[2] whose sufficient condition is generalized. An important contribution of Ref. 2 was to exhibit the power of time-dependent scattering theory for obtaining criteria of scattering equivalence. Unfortunately, the formalism of that reference has the essential limitation of being inapplicable to scattering phenomena involving bound-state fragments in the initial or final channels. Time-dependent methods are also used in the present publication, in which the approach of Refs. 1 and 2 is extended to multichannel scattering.

This extension is of direct relevance to nuclear physics, since it forms the basis of a new method[3] for the phenomenological study of many-body nuclear forces. The method has been successfully applied recently[4,5] to the bound trinucleon systems. However, the present paper is focused on the basic issues of scattering theory concerned and is understandable without any knowledge of its nuclear applications.

Let H and W^*HW be two unitarily equivalent, nonrelativistic, N-particle ($N \geq 2$) Hamiltonian operators acting in $L^2(\mathbb{R}^{3N})$. Typically, W^*HW involves nonlocal multiparticle interactions if $N \geq 3$, even if only local pair interactions occur in H.[3] In Sec. 2, only the case when the interactions in H are of short range is considered. The set of channel subspaces defined in that section for the case when the Hamiltonian of the N particles of interest is W^*HW is in bijective correspondence with the set of channel subspaces appropriate to the situation when their Hamiltonian is H. The definition is very natural physically. Given any pair of channel subspaces $\mathcal{H}_\alpha$, $\mathcal{H}_\beta$ of the latter type, it is proved under rather weak assumptions that the scattering amplitude for the process $\tilde{f}_\alpha \rightarrow \tilde{g}_\beta$ is equal to that for the process $f_\alpha \rightarrow g_\beta$ for each in state $f_\alpha \in \mathcal{H}_\alpha$ and each out state $g_\beta \in \mathcal{H}_\beta$. Here $\tilde{f}_\alpha(\tilde{g}_\beta)$ is the in (out) state into which $f_\alpha(g_\beta)$ is mapped by the bijective transformation in question. In Sec. 3, the theory of Sec. 2 is extended to a large class of nonrelativistic Hamiltonians H whose interactions can have both short-range and long-range parts. Suitable Coulombic potentials are included among these interactions. The generalization of the main results of Secs. 2 and 3 to short-range interactions with hard cores and the inclusion of spin and statistics is straightforward, but we have elected not to discuss these matters to avoid cumbersome notational complexities. It is also possible to formulate our results in the language of two-Hilbert space multichannel scattering theory[6] but, although elegant, this way of expressing our theory is less transparent physically and less convenient for applications than the standard multichannel scattering formulation employed in this paper and will not be mentioned any further here.

In Sec. 4, we consider two nontrivial examples of unitary operators W for which the above equality of corresponding scattering amplitudes holds for all entry and exit channel subspaces $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$. This equality holds in a large variety of cases in which H involves only short-range interactions or interactions which also have a long-range part. One of these examples is an operator W of the form $I+K$, where K is related to a compact operator in a certain L^2 space. The other example is an operator W which is a multichannel version of the transformations termed Bohm—Gross—Baker transformations in Ref. 1. Single-channel analogs of our examples have been extensively applied in nuclear physics.[7] Certain formal unitary operators used in Refs. 4 and 5 for $N=3$, when expressed in standard Hilbert space language, are essentially special cases of the multiparticle operators W investigated in Sec. 4.

An asymptotic theorem on the long-time evolution of wavepackets when certain long-range potentials act,

[a] This paper was completed when A. W. Sáenz was on sabbatical leave at the Department of Physics, Princeton University, Princeton, N. J.

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which is needed in Sec. 4, is stated and proved in the Appendix.

2. SCATTERING EQUIVALENCE WHEN THE INTERACTIONS IN H ARE OF SHORT RANGE

We consider $N \geq 2$ distinguishable particles with a Hamiltonian operator H, self-adjoint in $\mathcal{H} = L^2(\mathbb{R}^{3N})$. Typically, H is given in a formal sense by

$$-\sum_{i=1}^{N} (2m_i)^{-1} \Delta_{x_i} + \sum_{1 \leq i < j \leq N} V_{ij}, \qquad (2.1)$$

where the V_{ij} are pair interactions of short range, i.e., operators of multiplication in $\mathcal{H}$ by appropriate real functions $V_{ij}(x_i - x_j)$ on $\mathbb{R}^3$. For the general purposes of this section, detailed definitions of H and most of the other Hamiltonians therein are irrelevant.[8]

Let $D = \{C_1, \ldots, C_n\}$ be a partition of the N integers (particles) $\{1, \ldots, N\}$ into n "clusters" $C_1, \ldots, C_n$, where we always assume that $n \geq 2$. The number of integers in each C_l is denoted by n_l. Consider a linear mapping $x = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto (X_D, \xi_D)$ of $\mathbb{R}^{3N}$ onto itself having a Jacobian of absolute value unity. Here $X_D = (X_1, \ldots, X_n) \in \mathbb{R}^{3n}$ specifies the center-of-mass vectors $\mathbf{X}_l = \sum_{j \in C_l} m_j x_j / M_l$ of the particles in the various clusters, where $M_l = \sum_{j \in C_l} m_j$. For $n < N$, $\xi_D = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{N-n}) \in \mathbb{R}^{3(N-n)}$ determines their relative positions in each cluster, $\boldsymbol{\xi}_r$ $(r = 1, \ldots, N-n)$ being a linear combination of differences $\mathbf{x}_j - \mathbf{x}_k$ such that j and k are in the same cluster. We write $\xi^{(l)}$ for the vector $(\boldsymbol{\xi}_r) \in \mathbb{R}^{3(n_l - 1)}$ whose components ξ_r involve only differences of this kind with $j, k \in C_l$.

We associate with D the "free" Hamiltonian H_D, a self-adjoint operator in $\mathcal{H}$, formally obtained in the case (2.1) by omitting from this expression all V_{ij} linking different clusters. This property will be assumed in the sense that

$$(H_D f)(x) = (H_D^0 F)(X) \prod_{l=1}^{n} \varphi_l(\xi^{(l)}) + \sum_{\substack{l=1 \\ (n_l \geq 2)}}^{n} (h_l \varphi_l)(\xi^{(l)}) \prod_{l \neq l'=1}^{n} \varphi_{l'}(\xi^{(l')}) \qquad (2.2)$$

for all $f \in \mathcal{H}$ of the form $f(x) = F(X_D)\prod_{l=1}^{n} \varphi_l(\xi^{(l)})$, where $F \in D(H_D^0)$, $\varphi_l \in D(h_l)$ $(\varphi_l \equiv 1)$ if $n_l \geq 2$ $(n_l = 1)$. Here H_D^0 is the unique self-adjoint operator in $L^2(\mathbb{R}^{3n})$ which is an extension in $L^2(\mathbb{R}^{3n})$ of $-\sum_{l=1}^{n} (2M_l)^{-1} \Delta_{X_l}$ on $C_0^\infty(\mathbb{R}^{3n})$ and, for $n_l \geq 2$, h_l is the internal Hamiltonian of C_l, a self-adjoint operator in $L^2(\mathbb{R}^{3(n_l - 1)})$, formally constructed in the above example from the operator of type (2.1) involving only $i, j \in C_l$ by eliminating the center-of-mass motion.

Let D be such that either $n = N$ or that $n > N$ and that each h_l with $n_l \geq 2$ has a nonempty point spectrum. Let α be a channel consistent[9] with D, i.e., a pair $\alpha = \alpha_D = (b_D, D)$, where b_D is empty for $n = N$ and is otherwise a set $\{\psi_l, n_l \geq 2\}$ of eigenstates of h_l, one for each l with $n_l \geq 2$. The corresponding channel subspace $\mathcal{H}_\alpha$ is the set of all $\Psi_\alpha \in \mathcal{H}$ of the form

$$\Psi_\alpha(x) = F(X_D) \prod_{l=1}^{n} \psi_l(\xi^{(l)}), \qquad (2.3)$$

where $F \in L^2(\mathbb{R}^{3n})$ and $\psi_l \equiv 1$ for $n_l = 1$.

An immediate consequence of (2.2) and (2.3) will be needed, namely, that

$$(H_D \Psi_\alpha)(x) = [(H_D^0 + \epsilon_\alpha I) F](X_D) \prod_{l=1}^{n} \psi_l(\xi^{(l)}) \qquad (2.4)$$

for each of the states (2.3) with $F \in D(H_D^0)$. Here ϵ_α is the sum of the energy eigenvalues of all the bound states ψ_l when $n < N$ and is zero when $n = N$.

For each channel $\alpha = \alpha_D$, the Møller wave operators are given by

$$\Omega_\alpha^\pm = s - \lim_{t \to \pm\infty} \Omega_{\alpha, t} \qquad (2.5)$$

if the respective limit exists. Here

$$\Omega_{\alpha, t} = V_t^* U_{D, t} E_\alpha$$

for $-\infty < t < \infty$, with

$$U_{D, t} = \exp(-itH_D), \quad V_t = \exp(-itH) \qquad (2.6)$$

over this range of t,[10] E_α denoting the projection from $\mathcal{H}$ onto $\mathcal{H}_\alpha$.

When H is of the type (2.1), the existence of the wave operators (2.5) has been proved for general $N \geq 2$.[11] The more delicate property of asymptotic completeness has been proved for $N \geq 3$ only in a relatively small number of cases.[15] These proofs of existence and asymptotic completeness apply to appropriate short-range potentials.

We now consider the case when the Hamiltonian of the N particles of interest is

$$\tilde{H} = W^* H W,$$

where $W: \mathcal{H} \to \mathcal{H}$ is a unitary operator. We shall also need to define "free" Hamiltonians $\tilde{H}_D$ which will play a role with respect to $\tilde{H}$ analogous to that of the operators H_D in relation to H. Before defining the operators $\tilde{H}_D$, we shall impose certain conditions on W.

In addition to being unitary, it will be supposed in this section that W has the following two properties for each $D = \{C_1, \ldots, C_n\}$ $(n \geq 2)$:

(1) The limits

$$s\text{-}\lim_{t \to \pm\infty} U_{D, t}^* W U_{D, t} = W_D \qquad (2.7)$$

exist, their common value being a unitary operator W_D. In terms of the mapping $x \mapsto (X_D, \xi_D)$ we define $U_{D, t}: \mathcal{H} \to \mathcal{H}$ as the unique unitary operator such that

$$(U_{D, t} f)(x) = [\exp(-itH_D^0) F](X_D) \varphi(\xi_D) \qquad (2.8)$$

for functions $f \in \mathcal{H}$ of the form

$$f(x) = F(X_D) \varphi(\xi_D), \qquad (2.9)$$

with $F \in L^2(\mathbb{R}^{3n})$ and $\varphi \in L^2(\mathbb{R}^{3(N-n)})$ $(\varphi \equiv 1)$ if $n < N$ $(n = N)$.

(2) For each $f \in \mathcal{H}$ of the type specified immediately after (2.2),

$$(W_D f)(x) = F(X_D) \prod_{l=1}^{n} (w_l \varphi_l)(\xi^{(l)}), \qquad (2.10)$$

where w_l is a unitary operator on $L^2(\mathbb{R}^{3(n_l - 1)})$ when $n_l \geq 2$ and is unity when $n_l = 1$.

Examples of unitary operators W satisfying these two conditions for all D will be given in Sec. 4.[22] As far as we know, the problem of characterizing such operators in general is open at present.[23]

For a given D, (2.7) asserts that W_D arises from W

by making an infinite time displacement, toward the past or future, of the centers of the pertinent clusters. In a rough intuitive sense, this is the same as spatially displacing these centers infinitely far away from one another. Of course, the obvious precise statement of this equivalence is only true under suitable conditions.[24]

For each D, we define

$$\widetilde{H}_D = \widetilde{W}_D^* H_D W_D.$$

Because of (2.9), $\widetilde{H}_D \widetilde{f}$ satisfies an equation of the same product structure as (2.2), with the h_l replaced by $w_l^* h_l w_l$ and so forth, but with the same H_D^0 and F, where $\widetilde{f} = W_D^* f$, with f as stated immediately after (2.2). That is, loosely speaking, $\widetilde{H}_D$ contains no interactions between different clusters of the D considered and W_D does not affect the center-of-mass motion of the clusters.

Let $\alpha = \alpha_D$ be a channel in the sense defined previously. We define the subspace $\widetilde{\mathcal{H}}_\alpha$ of channel states of the transformed system corresponding to $\mathcal{H}_\alpha$ as the set of all states

$$\widetilde{\Psi}_\alpha = W_D^* \Psi_\alpha, \quad \Psi_\alpha \in \mathcal{H}_\alpha.$$

Since (2) holds for all D, $\widetilde{\Psi}_\alpha$ has the same type of product structure (2.3) as the corresponding Ψ_α.

The properties of the operators $\widetilde{H}_D$ and subspaces $\widetilde{\mathcal{H}}_\alpha$ stated in the proceding two paragraphs appear to be essential to construct a physically reasonable multichannel scattering theory when $\widetilde{H}$ is the Hamiltonian of the N particles. This was our motivation for imposing condition (2). Nevertheless, it is not necessary to assume that this condition is satisfied for any D in order to prove the main results of this section—those stated in Theorem 2.1.

For each channel $\alpha = \alpha_D$, the Møller wave operators appropriate to the case when the Hamiltonian of the N particles of interest is H are defined by

$$\widetilde{\Omega}_\alpha^\pm = s\text{-}\lim_{t\to\pm\infty} \widetilde{\Omega}_{\alpha,t} \tag{2.11}$$

when the respective limits exist. Here

$$\widetilde{\Omega}_{\alpha,t} = \widetilde{V}_t^* \widetilde{U}_{D,t} \widetilde{E}_\alpha, \tag{2.12}$$

where

$$\widetilde{U}_{D,t} = \exp(-it\widetilde{H}_D) = W_D^* U_{D,t} W_D,$$
$$\widetilde{V}_t = \exp(-it\widetilde{H}) = W^* V_t W,$$

and where $\widetilde{E}_\alpha$ denotes the projection from $\mathcal{H}$ onto $\mathcal{H}_\alpha$.

Let $\alpha = \alpha_D$ and $\beta = \beta_{D'}$ be channels consistent with the respective cluster decompositions D and D'. Provided the pertinent wave operators exist, we define partial scattering operators $S_{\beta\alpha}$ and $\widetilde{S}_{\beta\alpha}$ for the original and transformed system, respectively, in the conventional manner:

$$S_{\beta\alpha} = \Omega_\beta^{+*}\Omega_\alpha^-, \quad \widetilde{S}_{\beta\alpha} = \widetilde{\Omega}_\beta^{+*}\widetilde{\Omega}_\alpha^-.$$

(Theorem 2.1 entails that $\widetilde{S}_{\beta\alpha}$ exists if $S_{\beta\alpha}$ does.) We shall be interested in the case when the scattering amplitude from each in state of $\widetilde{\mathcal{H}}_\alpha$ to each out state of $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ is equal to the scattering amplitude between corresponding in and out states of $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$:

$$(\widetilde{g}_\beta, \widetilde{S}_{\beta\alpha}\widetilde{f}_\alpha) = (g_\beta, S_{\beta\alpha} f_\alpha), \quad \forall\ f_\alpha \in \mathcal{H}_\alpha, \quad \forall\ g_\beta \in \mathcal{H}_\beta. \tag{2.13}$$

Here

$$\widetilde{f}_\alpha = W_D^* f_\alpha \in \widetilde{\mathcal{H}}_\alpha, \quad \widetilde{g}_\beta = W_{D'}^* g_\beta \in \widetilde{\mathcal{H}}_\beta.$$

Theorem 2.1: Let $\alpha = \alpha_D$ be a channel consistent with a decomposition D and such that $\Omega_\alpha^\pm$ exist. Then $\widetilde{\Omega}_\alpha^\pm$ exist. Moreover, if $\beta = \beta_{D'}$ is a channel consistent with D', then (2.13) obtains for the channels α and β.

Remarks: Let $\Omega_\alpha^\pm$ exist for all channels and suppose that the familiar pairwise orthogonality property $R_\alpha^\pm \cap R_\beta^\pm = \{0\}$ holds for $\alpha \neq \beta$, where $R_\alpha^\pm$ are the ranges of $\Omega_\alpha^\pm$. Using, in particular, (2.17) in the proof of Theorem 2.1, one easily shows that asymptotic completeness holds for the ranges $\widetilde{R}_\alpha^\pm$ of the wave operators $\widetilde{\Omega}_\alpha^\pm$ iff it holds for the $R_\alpha^\pm$. That is, $\widetilde{R}_+ = \widetilde{R}_- = \mathcal{H}_{ac}(\widetilde{H})$ iff $R_+ = R_- = \mathcal{H}_{ac}(H)$, where $R_\pm = \oplus_\alpha R_\alpha^\pm$, $\widetilde{R}_\pm = \oplus_\alpha \widetilde{R}_\alpha^\pm$, the direct sums running over all channels and $\mathcal{H}_{ac}(H)$, $\mathcal{H}_{ac}(\widetilde{H})$ being the subspaces of absolute continuity of H, $\widetilde{H}$, respectively.

A remark similar to that in the penultimate sentence can be made about the scattering systems considered in Sec. 3.

Proof of Theorem 2.1[25]: We first observe that

$$\Omega_{\alpha,t} = \exp(-i\epsilon_\alpha t)\, V_t^* U_{D,t} E_\alpha,$$
$$\widetilde{\Omega}_{\alpha,t} = \exp(-i\epsilon_\alpha t)\, \widetilde{V}_t^* U_{D,t} \widetilde{E}_\alpha, \tag{2.14}$$

as follows directly, in particular from the appropriate definitions, (2.4), and the facts that $W_D \widetilde{\mathcal{H}}_\alpha = \mathcal{H}_\alpha$ and that W_D commutes with $U_{D,t}$ and is unitary. This commutativity follows from the assumption that W_D exists and is equal to the limits (2.7), together with the fact that $\{U_{D,t}, -\infty < t < \infty\}$ is a unitary group.

Using (2.12), (2.14), and the identity

$$W_D \widetilde{E}_\alpha = E_\alpha W_D, \tag{2.15}$$

we can write

$$\widetilde{\Omega}_{\alpha,t} = W^* \Omega_{\alpha,t} W_D + \exp(-i\epsilon_\alpha t) \times W^* V_t^* (W U_{D,t} - U_{D,t} W_D)\widetilde{E}_\alpha. \tag{2.16}$$

In view of (2.7), the unitary of $U_{D,t}$ and the uniform boundedness of $\exp(-i\epsilon_\alpha t)\ W^* V_t^*$, the second term on the rhs of (2.16) converges strongly to zero as $t \to \pm\infty$. From this fact, the assumed existence of $\Omega_\alpha^\pm$ and (2.14), we infer that $\widetilde{\Omega}_\alpha^\pm$ exist and are given by

$$\widetilde{\Omega}_\alpha^\pm = W^* \Omega_\alpha^\pm W_D \tag{2.17}$$

under the present hypotheses.

We now prove that (2.13) obtains for the channels $\alpha = \alpha_D$, $\beta = \beta_{D'}$. Using, in particular, (2.15) and its analog for channel β, it is easily seen that (2.13) is equivalent to

$$S_{\beta\alpha} = W_{D'} \widetilde{S}_{\beta\alpha} W_D^*. \tag{2.18}$$

Expressing $\widetilde{S}_{\beta\alpha}$ in terms of the operators $\widetilde{\Omega}_\alpha^-$ and $\widetilde{\Omega}_\beta^+$ as given by (2.17) and its analog for channel β and invoking the unitarity of the relevant operators, we find that (2.18) is satisfied for the channels α and β considered.

3. GENERALIZATION TO THE PRESENCE OF LONG-RANGE INTERACTIONS IN H

In this section, we find it necessary to be rather specific about the Hamiltonian H of the N particles. This is due primarily to the complications introduced by the occurrence of long-range interactions. H is defined here as the unique self-adjoint extension in $\mathcal{H} = L^2(\mathbb{R}^{3N})$ of a differential operator of the form (2.1) on $C_0^\infty(\mathbb{R}^{3N})$, with each V_{ij} $(1 \le i < j \le N)$, an operator of multiplication by a function $V_{ij}(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i, \mathbf{x}_j \in \mathbb{R}^3)$, with

$$V_{ij}(\cdot) = V_{ij}^S(\cdot) + V_{ij}^L(\cdot). \tag{3.1}$$

The short-range and long-range parts of $V_{ij}(\cdot)$, $V_{ij}^S(\cdot)$ and $V_{ij}^L(\cdot)$, respectively, are real functions on $\mathbb{R}^3$. We also suppose that there exist three positive constants, δ, C, and ρ, with $0 < \rho \le 1$, such that

$$(1 + |x|)^{1+\delta} \, |V_{ij}^S(x)| \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3), \tag{3.2a}$$

$$|\nabla^p V_{ij}^L(x)| \le C(1 + |x|)^{-(p+\rho)}, \tag{3.2b}$$

for all positive integers p.

For every D, H_D and the h_l are as defined for systems of the type (2.1) by the third paragraph of Sec. 2 and the second sentence of Ref. 8. In particular, H_D is the unique self-adjoint extension in $\mathcal{H}$ of the differential operator (2.1) on $C_0^\infty(R^{3N})$ for the V_{ij} in (3.1). Naturally, the terminology $\alpha = \alpha_D$ and $\mathcal{H}_\alpha$ will be understood now in terms of the latter operators h_l. Hence (2.2) and (2.4) are satisfied for elements of $\mathcal{H}$ of the respective indicated types in terms of the operators H_D and h_l of this section.

Consider a cluster decomposition $D = \{C_1, \ldots, C_n\}$. For every nonnegative integer r, $\Gamma_{D,t}^{(r)}$ is an operator of multiplication in the momentum-space representation by $L^2(\mathbb{R}^{3n})$ by a function $\Gamma_{D,t}^{(r)}(P)$ defined recursively by

$$\begin{aligned} &\Gamma_{D,t}^{(0)}(P) = 0, \\ &\Gamma_{D,t}^{(r)}(P) = \int_0^t \mathcal{V}_L(sM^{-1}P + \nabla_P \Gamma^{(r-1)}(P))\, ds, \\ &r = 1, 2, \ldots, \end{aligned} \tag{3.3}$$

for $P = (\mathbf{P}_1, \ldots, \mathbf{P}_n) \in \mathbb{R}^{3n}$. Here $sM^{-1}P = (sM_1^{-1}\mathbf{P}_1, \ldots, sM_n^{-1}\mathbf{P}_n)$, with $M_l = \sum_{j \in C_l} m_j$ as before, $\nabla_P = (\nabla_{P_1}, \ldots, \nabla_{P_n})$, and $\mathcal{V}_L$ is the following real function on R^{3n}:

$$\mathcal{V}_L(\mathbf{X}_1, \ldots, \mathbf{X}_n) = \sum_{1 \le \lambda < \mu \le n} \sum_{\substack{i \in C_\lambda \\ j \in C_\mu}} V_{ij}^L(\mathbf{X}_\lambda - \mathbf{X}_\mu). \tag{3.4}$$

At each such r, let $G_{D,t}^{(r)}$ be the unique bounded self-adjoint operator for which

$$(G_{D,t}^{(r)} f)(x) = (\Gamma_{D,t}^{(r)} F)(X_D)\varphi(\xi_D) \tag{3.5}$$

when f is of the product form (2.9), with F and φ as specified immediately after the latter equation.

For every channel $\alpha = \alpha_D$ consistent with D, we define modified wave operators:

$$\Omega_\alpha^\pm = \underset{t \to \pm\infty}{\text{s-lim}}\, \Omega_{\alpha,t} \tag{3.6}$$

if the respective limits exist, where now

$$\Omega_{\alpha,t} = V_t^* U'_{D,t} E_\alpha.$$

Here E_α is the projection of $\mathcal{H}$ onto $\mathcal{H}_\alpha$, interpreted in the present sense, V_t is given by (2.6) in terms of the H of this section, and

$$U'_{D,t} = U_{D,t} \exp(-iG_{D,t}^{(m)}),$$

where $U_{D,t}$ is defined by (2.6), with H_D understood as in this section. We denote by m the smallest positive integer greater than $\rho^{-1} - 1$.

It can be shown that the wave operators (3.6) exist for all channels when the conditions (3.1)–(3.2b) on the V_{ij} are satisfied. Actually, we have proved this under much weaker hypotheses on the V_{ij}^L, with m understood in a generalized sense.[26]

We again consider a unitary operator $W: \mathcal{H} \to \mathcal{H}$. In the present section, W_D is a unitary operator having the properties (1′) and (2) for all D, where the latter property was stated in Sec. 2 and the former property is as follows:

(1′) The equations

$$W_D = \underset{t \to \pm\infty}{\text{s-lim}}\, \mathcal{U}'^*_{D,t} W \mathcal{U}'_{D,t}, \tag{3.7}$$

hold. Here $\mathcal{U}'_{D,t}: \mathcal{H} \to \mathcal{H}$ is the unitary operator

$$\mathcal{U}'_{D,t} = \mathcal{U}_{D,t} \exp(-iG_{D,t}^{(m)}), \tag{3.8}$$

with $\mathcal{U}_{D,t}$ as in (2.8).

Since $\mathcal{U}'_{D,t}$ controls the long-time evolution of the centers of mass of the clusters of the D of interest for the interactions now being considered (Lemma A.1 of the Appendix), (3.7) can be interpreted physically, *grosso modo*, in a manner similar to that mentioned in connection with (2.7).[27] The motivation for imposing condition (2) here is the same as in Sec. 2.

We define $\tilde{H}$, $\tilde{H}_D$, $\tilde{\mathcal{H}}_\alpha$, and $\tilde{E}_\alpha$ in this section just as in the previous one, but of course with H and H_D understood in the sense of the present section. When $\tilde{H}$, instead of H, is the Hamiltonian of the relevant N particles, the modified wave operators corresponding to each $\alpha = \alpha_D$ are defined by (2.11) when they exist, but with

$$\tilde{\Omega}_{\alpha,t} = \tilde{V}_t^* U'_{D,t} \tilde{E}_\alpha.$$

Here $\tilde{V}_t$ is as in (2.6) in terms of the present $\tilde{H}$ and

$$\tilde{U}'_{D,t} = W_D^* U'_{D,t} W_D.$$

Naturally, the partial scattering operators $S_{\beta\alpha}$ and $\tilde{S}_{\beta\alpha}$ of this section are given by the same formulas of Sec. 2, with Ω_β^+, Ω_α^-, $\tilde{\Omega}_\beta^-$, and $\tilde{\Omega}_\alpha^-$ signifying the respective modified wave operators of the previous paragraph.

Theorem 3.1: Let $\alpha = \alpha_D$ be a channel consistent with the decomposition D and let $\Omega_\alpha^\pm$ exist. Then $\Omega_\alpha^\pm$ exist. In addition, let $\beta = \beta_{D'}$ be a channel consistent with the decomposition D'. Then (2.13), interpreted in the sense of this section, holds for the channels α and β.

Proof: A proof similar to that of Theorem 2.1, with $\mathcal{U}'_{D,t}$ playing the role of $\mathcal{U}_{D,t}$ in the earlier proof, serves to establish the present theorem. In particular, one employs equations of the same structure as (2.14), but with $\mathcal{U}_{D,t}$ replaced by $\mathcal{U}'_{D,t}$. To derive these equations, one invokes, among other properties, the commutativity of W_D with $\mathcal{U}'_{D,t}$, which follows by using condition (2), (3.5), (3.8), and the corresponding commutativity property of $\mathcal{U}_{D,t}$.

4. EXAMPLES OF OPERATORS W FOR WHICH (2.13) HOLDS FOR ALL CHANNELS

These examples are of two types. In subsection A of this section, we consider a unitary operator $W=W_1=I+K$, where K is defined in terms of a certain compact operator. In subsection B, we discuss a unitary operator $W=W_2$ of the Bohm–Gross–Baker type. From results of this section, it follows for all D that W_1 and W_2 have property (1′) of Sec. 3, and therefore (1) of Sec. 2, as well as property (2) of Sec. 2.

Suppose that one of the following cases obtains: (i) H and every H_D are self-adjoint operators, with each H_D satisfying (2.2) for the indicated functions f and with the wave operators (2.5) existing for all channels; (ii) H and every H_D are as prescribed in Sec. 3, with the p-independent constant ρ in (3.2b) obeying $\frac{1}{2}<\rho<1$. *Then* (2.13), *understood in either the sense of Sec.* 2 *or Sec.* 3, *holds for all channels* α *and* β *when* $W=W_i$ $(i=1,2)$. This follows by applying Theorems 2.1 and 3.1 to W_1 and W_2, the application being legitimate in view of the properties of these operators stated in the penultimate sentence of the previous paragraph and of the assumptions made in the first sentence of the present paragraph.

It is physically reasonable to demand that the independence of the relative motion from the center-of-mass motion, which holds for typical nonrelativistic N-particle systems with translationally invariant interactions under suitable technical assumptions, should also hold for the corresponding N-particle transformed systems. More precisely, let the Hamiltonian H of the original system satisfy the equation

$$(Hg)(x)=(H_0G)(\mathbf{X})\chi(\eta)+G(\mathbf{X})(h\chi)(\eta)$$

for each $g\in D(H)$ of the form $g(x)=G(\mathbf{X})\chi(\eta)$, with $G\in D(H_0)$, $\chi\in D(h)$, $H_0(h)$ being the usual self-adjoint Hamiltonian in $L^2(R^3)(L^2(R^{3(N-1)}))$ governing the latter (former) motion. Here a transformation $x=(\mathbf{x}_1,\ldots,\mathbf{x}_N)\mapsto(\mathbf{X},\eta)$ with Jacobian of absolute value unity is understood, where $\eta=(\eta_1,\ldots,\eta_{N-1})\in\mathbb{R}^{3(N-1)}$, each η_k being a linear combination of differences $\mathbf{x}_i-\mathbf{x}_j$. Plainly, for all such g an equation of the same structure will obtain for $\tilde{H}\tilde{g}$, where $\tilde{H}=W^*HW$ and $\tilde{g}=W^*g$ for the unitary W of interest, the same H_0 and G appearing in both equations, if

$$(Wg)(x)=G(\mathbf{X})(w\chi)(\eta) \tag{4.1}$$

for every such g, with w a unitary operator. The operator W_1 in subsection A evidently has this property and the operator W_2 in subsection B can be specialized to possess it. However, the property in question is irrelevant with respect to whether (2.13) holds for any channel α and β, either in the sense of Sec. 2 or Sec. 3.

A. Example $W=W_1$

Let us express the position vectors $\mathbf{x}_1,\ldots,\mathbf{x}_N$ of the N particles in terms of the variables $\mathbf{X}$, θ_i $(i=1,\ldots,3N-4)$, R. Here $\mathbf{X}$ is the center-of-mass vector $\sum_{j=1}^N m_j\mathbf{x}_j/\sum_{j=1}^N m_j$ of the particles and the $\theta_i\in I_i$ and $R=(\sum_{1\le i<j\le N}c_{ij}\times|\mathbf{x}_i-\mathbf{x}_j|^2)^{1/2}\in[0,\infty)$ are the so-called hyperspherical angles and the hyperspherical radius, respectively. The c_{ij} are positive constants and each I_i $(i=1,\ldots,3N-4)$ is a finite interval. We set $I_{3(N-1)}=[0,\infty)$ for convenience. The $3(N-1)$ variables θ_i, R are such that each coordinate difference $\mathbf{x}_j-\mathbf{x}_k$ can be expressed solely in terms of them.[28]

More precisely, we consider a mapping $\mathcal{k}:(\mathbf{X},y,z)\mapsto x=(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ of $\mathbb{R}^3\times\mathcal{M}\times\mathcal{N}$ into $\mathbb{R}^{3N}$, where $\mathcal{M}=\times_{i\in B}I_i$, A being a nonempty subset of $\{1,\ldots,3(N-1)\}$ and $B=\{1,\ldots,3(N-1)\}\setminus A$. In addition, we suppose that $3(N-1)\in B$. The intervals I_i in the cartesian products $\mathcal{M}$ and $\mathcal{N}$ are ordered in a fixed, but arbitrary manner. If we regard, as we shall, $\mathcal{M}$ and $\mathcal{N}$ as subspaces of appropriate linear spaces $\mathbb{R}^p$ and $\mathbb{R}^q$, normed in the usual way and with p and q positive integers such that $p+q=3(N-1)$, it is clear that $\mathcal{M}$ is bounded and $\mathcal{N}$ is unbounded. The boundedness of $\mathcal{M}$ will play an essential role in this subsection.

For any positive values of the c_{ij}, there are many ways of selecting the θ_i and I_i so that $\mathcal{k}$ has the following properties, which will be assumed to hold henceforth. First, $\mathcal{k}(\mathcal{O})$ is a one–one differentiable mapping onto an open subset $\mathcal{O}$ of $\mathbb{R}^{3N}$ differing at most from $\mathbb{R}^{3N}$ by a set of measure zero, where $\mathcal{O}$ is obtained from $R^3\times\mathcal{M}\times\mathcal{N}$ by deleting the endpoints of all the I_i. Second, the absolute value of the Jacobian of $\mathcal{k}$ at any $(\mathbf{X},y,z)\in\mathcal{O}$ equals $\sigma(z)$, where σ is a real-valued function on $\mathcal{M}$ which is positive at each of the latter z values and bounded on each bounded subset of $\mathcal{N}$.

Hyperspherical coordinates *per se* are of little interest here. They were introduced merely as a simple, convenient way of defining a mapping $\mathcal{k}$ with the stated properties.

Let $W_1:\mathcal{H}\to\mathcal{H}$ be a unitary operator of the form

$$W_1=I+K, \tag{4.2}$$

$K:\mathcal{H}\to\mathcal{H}$ being a bounded operator such that

$$(Kf)(x)=\sigma^{-1/2}(z)(k\mathcal{J}(\mathbf{X},y,\cdot))(z), \tag{4.3}$$

for $f\in\mathcal{H}$, where

$$\sigma^{-1/2}(z)\,\mathcal{J}(\mathbf{X},y,z)=f(x) \tag{4.4}$$

and where, of course, x is the image of (X,y,z) under $\mathcal{k}$. The assumption that $f\in\mathcal{H}$ together with the stated properties of $\mathcal{k}$ entail, via a change of variables in the pertinent integral, that $\mathcal{J}(X,y,\cdot)\in L^2(\mathcal{N})$ a.e. The symbol k in (4.3) denotes a compact operator from $L^2(\mathcal{N})$ into itself, but it is easy to see that K is not compact.[29] The factor $\sigma^{-1/2}(z)$ has been introduced for convenience.

Consider the simple example in which

$$k=[\exp(i\alpha)-1]p,$$

where α is a real number and p is a projection operator from $L^2(\mathcal{N})$ onto a one-dimensional subspace spanned by $\Phi\in L^2(\mathcal{N})$, with

$$\|\Phi\|_{L^2(\mathcal{N})}=1. \tag{4.5}$$

Hence

$$K=[\exp(i\alpha)-1]P$$

in this example, $P: \mathcal{H} \to \mathcal{H}$ being a projection of infinite-dimensional range, obviously given by

$$(Pf)(x) = \sigma^{-1/2}(z)(\Phi, \mathcal{J}(\mathbf{X}, y, \cdot))_{L^2(\mathcal{N})}\Phi(z) \tag{4.6}$$

for each $f \in \mathcal{H}$. That P is a projection follows easily, in particular, by changing variables in the integral of interest and using (4.5) and Fubini's theorem. [Arguments of this type will be used henceforth in this subsection without explicit comment.] Hence the operator $W_1 = I + [\exp(i\alpha) - 1]P = \exp(i\alpha P)$ appropriate to this special case is unitary. For future use, we remark that

$$\|Pf\|^2 = \int_{\mathbb{R}^3 \times \mathcal{M}} |(\Phi, \mathcal{J}(\mathbf{X}, y, \cdot))_{L^2(\mathcal{N})}|^2 d\mathbf{X}\, dy < \infty \tag{4.7}$$

at each such f, $\|\cdot\|$ standing as before for the $\mathcal{H} = L^2(\mathbb{R}^{3N})$ norm.

Theorem 4.1: For each D, let $\mathcal{U}'_{D,t}$ be the operator defined by (3.8) for $m=1$, where each function $V^L_{ij}(\cdot)$ $(1 \le i < j \le N)$ occurring implicitly in this definition is a real function on $\mathbb{R}^3$ satisfying (3.2b) for $p=1$ with a constant $\rho > \frac{1}{2}$. Let $W_1: \mathcal{H} \to \mathcal{H}$ be the unitary operator (4.2), with K as defined in the paragraph containing (4.2). Then (3.7) [and therefore (2.7)] holds for all D with $W = W_1$ and $W_D = I$,

$$\underset{t \to \pm\infty}{\text{s-lim}}\, \mathcal{U}'^{*}_{D,t} W_1 \mathcal{U}'_{D,t} = I.$$

Remarks: Let W'_1 be a unitary operator of the form $I + K'$, where $(K'f)(x) = (\kappa H(\mathbf{X}, \cdot))(\eta)$ for each $f \in \mathcal{H}$. Here $H(\mathbf{X}, \eta) = f(x)$, the linear transformation $x = (\mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto (\mathbf{X}, \eta)$ mentioned in the paragraph containing (4.1) being understood and $\kappa: L^2(\mathbb{R}^{3(N-1)}) \to L^2(\mathbb{R}^{3(N-1)})$ is a compact operator. Then the conclusion of Theorem 4.1 holds with W_1 replaced by W'_1 as follows by an approach analogous to that used in the proof of this theorem below. When the action of K' on $f \in \mathcal{H}$ is expressed in terms of suitable hyperspherical coordinates, an equation similar to (4.3) is obtained, as expected, but with k in (4.3) replaced by a compact operator from $L^2(\mathcal{N}')$ into $L^2(\mathcal{N}')$, where $\mathcal{N}' = \times_{i=1}^{3(N-1)} I_i$. In contrast to the usefulness of these coordinates elsewhere in this subsection, this is merely a cumbersome restatement of the above definition of K'.

Proof of Theorem 4.1: In the proof, K and k will be as defined in the paragraph containing (4.2). The assertion of the theorem is clearly equivalent to the statement that

$$\underset{t \to \pm\infty}{\text{s-lim}}\, K\mathcal{U}'_{D,t} = 0 \tag{4.8}$$

obtains for each D. To prove this statement, we make a series of simplifying remarks.

Since k is compact, there exists for each $\epsilon > 0$ a finite operator sum $\sum_{i=1}^r a_i p_i$ such that

$$\|k - \sum_{i=1}^r a_i p_i\|_{L^2(\mathcal{N})} < \epsilon, \tag{4.9}$$

the a_i being complex constants and each p_i a projection from $L^2(\mathcal{N})$ onto its one-dimensional subspace spanned by Φ_i, with $\|\Phi_i\|_{L^2(\mathcal{N})} = 1$. One easily sees that (4.9) is true when k, p_i, and $\|\cdot\|_{L^2(\mathcal{N})}$ are replaced by K, P_i, and $\|\cdot\|$ $(i = 1, \ldots, r)$, where P_i is defined by (4.6), with Φ_i replacing Φ. With the aid of this last result and of the unitarity of each $\mathcal{U}'_{D,t}$, we conclude that

$$\|K\mathcal{U}'_{D,t}f\| \le \sum_{i=1}^r |a_i| \; \|P_i\mathcal{U}'_{D,t}f\| + \epsilon\|f\|$$

for $f \in \mathcal{H}$ and all D. Whence (4.8) holds if

$$\underset{t \to \pm\infty}{\text{s-lim}}\, P\mathcal{U}'_{D,t} = 0 \tag{4.10}$$

for the same D when P is given by (4.6). From now on, we fix Φ, and therefore P.

It is enough to prove (4.10) for every D in the case when Φ, besides satisfying (4.5) is a bounded function of bounded support. To see this, observe that for each $\epsilon > 0$ there exists a $\Phi' \in L^2(\mathcal{N})$, bounded and of bounded support, and such that $\|\Phi - \Phi'\|_{L^2(\mathcal{N})} < \epsilon$ and $\|\Phi'\|_{L^2(\mathcal{N})} = 1$. Define P' by (4.5), with Φ' replacing Φ. Then elementary results of integration theory, together with (4.5), the last two properties of Φ', and the unitarity of each $\mathcal{U}'_{D,t}$ yield

$$\|P\mathcal{U}'_{D,t}f\| < \|P'\mathcal{U}'_{D,t}f\| + 2\epsilon\|f\|$$

for $f \in \mathcal{H}$ and all D, from which the asserted sufficiency follows. Henceforth, Φ will be a bounded function of compact support.

In the remainder of this proof, $D = \{C_1, \ldots, C_n\}$ will be fixed and f will be a fixed function of the product form (2.9), with $F \in \mathcal{S}_D$ and φ a bounded function in $L^2(R^{3(N-n)})$ for $n < N$ and $\varphi \equiv 1$ for $n = N$. The space $\mathcal{S}_D$ is defined in the first sentence of the paragraph following Eq. (A1) of the Appendix. Since $\|P\mathcal{U}'_{D,t}\| = 1$ and since the closed span of the elements of the type (2.8) just specified equals $\mathcal{H}$, it suffices to show that

$$\lim_{t \to \pm\infty} \|P\mathcal{U}'_{D,t}f\| = 0. \tag{4.11}$$

An important simplification in the proof arises from the fact that, under the present hypotheses, Lemma A.1 entails that

$$\lim_{|t| \to \infty} \|(\mathcal{U}'_{D,t} - \mathcal{Z}_{D,t})f\| = 0, \tag{4.12}$$

$\mathcal{Z}_{D,t}$ being defined by (A3) of the Appendix for the f, F, and φ presently under consideration. In (A3) and elsewhere, $\hat{F}$ denotes the $L^1(R^{3n})$ Fourier transform of F.

Because of (4.12) and the boundedness of P, (4.11) will follow if it is proved that

$$\lim_{|t| \to \infty} \|P\mathcal{Z}_{D,t}f\| = 0. \tag{4.13}$$

We proceed to do this.

Let $\mathcal{K} = \text{supp}\Phi$. Using, in particular, (4.7), (A3) interpreted in the present sense, the boundedness of φ, the boundedness and support properties of Φ, the fact that σ is bounded on the compact subset $\mathcal{K} \subset \mathbb{R}^q$, and making the change

$$\mathbf{u} = t^{-1}\mathbf{X}$$

of integration variable for $t \neq 0$, we find readily that

$$\|P\mathcal{Z}_{D,t}f\|^2 \le \text{const}\,|t|^{-3(n-1)} \times \int_{\mathbb{R}^3 \times \mathcal{N}} \Big[\int_{\mathcal{K}} |\hat{F}(Y_t(u, y, z))|\, dz\Big]^2 d\mathbf{u}\, dy \tag{4.14}$$

at every such t. The vector-valued function $Y_t(u, y, z)$ is $t^{-1}Y = (t^{-1}M_1\mathbf{X}_1, \ldots, t^{-1}M_n\mathbf{X}_n)$ expressed in terms of the indicated vector variables and is thus of the form $(M_1\mathbf{u} + t^{-1}\boldsymbol{\beta}_1(y, z), \ldots, M_n\mathbf{u} + t^{-1}\boldsymbol{\beta}_n(y, z))$, where, for each

$l=1,\ldots,n$, $\boldsymbol{\beta}_l:\mathcal{M}\times\mathcal{N}\to\mathbb{R}^3$ is t independent with $|\boldsymbol{\beta}_l(v,z)|$ bounded on each bounded subset of $\mathcal{M}\times\mathcal{N}$.

Since F is of compact support and $|\boldsymbol{\beta}_l(v,z)|$ is bounded on $\mathcal{M}\times\mathcal{K}$ for each l, it is easy to see that, for any preassigned $t_1>0$, we can replace $\mathbb{R}^3$ in the domain of integration in (4.14) by some compact, t-independent subset of $\mathbb{R}^3$ when $t\geq t_1$. Thus the integral on the rhs of (4.14) is bounded by some finite, t-independent number for $t\geq t_1$. It follows that

$$\|PZ_{D,t}f\|\leq \text{const}\,|t|^{-3(n-1)/2}$$

at each such t and therefore that (4.13) holds.

B. Example $W=W_2$[30]

For each $D=\{C_1,\ldots,C_n\}$, it will again be necessary to consider the transformation $x=(\mathbf{x}_1,\ldots,\mathbf{x}_N)\mapsto(X_D,\xi_D)$. For notational convenience, we shall denote $X_D=(\mathbf{X}_1,\ldots,\mathbf{X}_n)$ and $\xi_D=(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{N-n})$ by X and ξ, respectively, in this subsection. We introduce mappings h and h_D having the following properties for every D:

(a) $h:\mathbb{R}^{3N}\to\mathbb{R}^{3N}$ and $h^D:\mathbb{R}^{3N}\to\mathbb{R}^{3N}$ are injective and continuously differentiable, and their respective Jacobians $J(x)$ and $J_D(x)$ are nonvanishing on $\mathbb{R}^{3N}$. Moreover, h is independent of D and h^D is the identity map on $\mathbb{R}^{3N}$ for $n=N$. We write $h(x)=(\mathbf{h}_1(x),\ldots,\mathbf{h}_N(x))$ and $h^D(x)=(\mathbf{h}_1^D(x),\ldots,\mathbf{h}_N^D(x))$ for each $x=(\mathbf{x}_1,\ldots,\mathbf{x}_N)\in\mathbb{R}^{3N}$.

(b) If $n<N$, h^D leaves each $\mathbf{X}_l$ invariant, i.e.,

$$\sum_{j=1}^N m_j\,\mathbf{h}_j^D(x)=\sum_{j=1}^N m_j\mathbf{x}_j,\quad x\in\mathbb{R}^{3N}. \tag{4.15}$$

(c) If $i\in C_l\subset D$ and $n_l\geq 2$, then $\mathbf{h}_i^D(x)-\rho$ depends on x only through $\mathbf{x}_j-\mathbf{x}_k$ with $j,k\in C$. We can thus write $\rho_i^D(\xi)=\mathbf{h}_i^D(x)-\mathbf{X}_i$ for each i and x, where the lhs is independent of X [in this notation, which is abusive for the case $n=N$, each $\rho_i^D(\xi)$ is identically zero by (a) and (c) in this case].

(d) Define $\rho_i(X,\xi)=\mathbf{h}_i(x)-\mathbf{x}_i$ for each i and x. If $n<N$, then at each $(X,\xi)\in\mathbb{R}^{3N}\times\mathbb{R}^{3(N-n)}$,

$$\lim_{\nu(\mathbf{X})\to\infty}|\rho_i(X,\xi)-\rho_i^D(\xi)|=0,$$
$$\lim_{\nu(\mathbf{X})\to\infty}|\partial\rho_i(X,\xi)/\partial X_{l,p}|=0, \tag{4.16}$$
$$\lim_{\nu(\mathbf{X})\to\infty}|\partial\rho_i(X,\xi)/\partial\xi_{r,p}-\partial\rho_i^D(\xi)/\partial\xi_{r,p}|=0,$$

for all $i=1,\ldots,N$, $l=1,\ldots,n$, $r=1,\ldots,N-n$, $p=1,2,3$, where $\nu(X)=\min_{1\leq l<m\leq n}(\mathbf{X}_l-\mathbf{X}_m)$ and p labels the pertinent Cartesian components. If $n=N$, then the first two Eqs. (4.16) are satisfied at each $X\in\mathbb{R}^{3N}$ over the stated range of indices.

(e) At every i, r, l, and p in (4.16), $\rho_i(X,\xi)$, $\rho_i^D(\xi)$, and their partial derivatives with respect to $X_{l,p}$ and $\xi_{r,p}$ are bounded over $\mathbb{R}^{3n}\times\mathbb{R}^{3(N-n)}$ when $n\leq N$. This statement holds for the partial derivatives with respect to $X_{l,p}$ with $\mathbb{R}^{3n}\times\mathbb{R}^{3(N-n)}$ replaced by $\mathbb{R}^{3N}$ when $n=N$.

Remarks:

1. For any given D, (a), (b), and (c) jointly guarantee that property (2) of Sec. 2 holds when W_D is replaced by $W_{2,D}$ in (2.10) and that $W_{2,D}$ [defined in (4.18) below] is the unit operator in $L^2(\mathbb{R}^{3(N-n)})$ when D corresponds to the free channel. However, it is not necessary for (b) to hold in order for Theorem 4.2 to be true. Assumption (d) states for each D that $h(x)$ and $h_D(x)$, as well as their derivatives, agree at large intercluster separations $[\nu(X)\to\infty]$. A number of the hypotheses about h and the h_D have been made to avoid annoying complications. Specifically, the conditions on differentiability and the nonvanishing of the pertinent Jacobians can be slightly relaxed.

2. It is perhaps not obvious that mappings h and h_D with all of the above properties exist. Here is an example and many more can be easily devised. For each $x\in\mathbb{R}^{3N}$ and $D=\{C_1,\ldots,C_n\}$, let

$$\mathbf{h}_i(x)=\mathbf{x}_i+\sum_{i\neq j=1}^N\left(\frac{m_j}{m_i}\right)^{1/2}g_{ij}(|\mathbf{x}_{ij}|^2)\mathbf{x}_{ij},\quad i=1,\ldots,N,$$

$$\mathbf{h}_i^D(x)=x_i+\sum_{i\neq j\in C_l}\left(\frac{m_j}{m_i}\right)^{1/2}g_{ij}(|x_{ij}|^2)\mathbf{x}_{ij},\quad i\in C_l,\ l=1,\ldots,n,$$

where, of course, the summation in the expression for $\mathbf{h}_i^D(x)$ is to be omitted if $i\in C_l$ with $n_l=1$. Here m_i is again the mass of the ith particle and $\mathbf{x}_{ij}=\mathbf{x}_i-\mathbf{x}_j$. For each $i\neq j$ ($i,j=1,\ldots,N$), $g_{ij}:[0,\infty)\to\mathbb{R}$ has the properties: $g_{ij}=g_{ji}$, $g_{ij}\in C^1([0,\infty))$, $g_{ij}(r)=o(r^{1/2})$ as $r\to\infty$, and $2r|dg_{ij}(r)/dr|+|g_{ij}(r)|\leq c$ on $[0,\infty)$, c being a positive constant. Invoking, in particular, the contraction mapping principle in $\mathbb{R}^{3N}$, it follows that (a) is satisfied by this example for every D if c is small enough. It is easy to verify that the remaining conditions on h and the h_D are also satisfied for all D.

For every $\psi\in\mathcal{H}$ and D, the operators W_2 and $W_{2,D}$ are defined by

$$(W_2\psi)(x)=\sigma(x)\,\psi(h(x)), \tag{4.17}$$

$$(W_{2,D}\psi)(x)=\sigma_D(x)\,\psi(h_D(x)), \tag{4.18}$$

where $\sigma(x)=|J(x)|^{1/2}$ and $\sigma_D(x)=|J_D(x)|^{1/2}$.[30]

Using assumption (a) and elementary theorems of integration theory, one can show that W_2 and $W_{2,D}$ are unitary for all D. Notice that W_2 satisfies (4.1) for each g of the indicated form if h fulfills the following additional conditions: (4.15) holds with h_D replaced by h and each $\mathbf{h}_i(x)-\mathbf{x}_i$ depends on $x=(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ only through differences $\mathbf{x}_i-\mathbf{x}_j$.

Theorem 4.2: Define each $\mathcal{U}'_{D,t}$ as in Theorem 4.1. Then (3.7) holds for all D with $W=W_{2,D}$:

$$\operatorname*{s-lim}_{t\to\pm\infty}\mathcal{U}'^{*}_{D,t}W_2\mathcal{U}'_{D,t}=W_{2,D}.$$

Proof: In it, $D=\{C_1,\ldots,C_n\}$ will be considered fixed and we will assume that $n<N$. The case $n=N$ yields to analogous, simpler arguments.

The operators $W_{2,D}$ and $\mathcal{U}'_{D,t}$ commute, since $W_{2,D}\mathcal{U}'_{D,t}$ and $\mathcal{U}'_{D,t}W_{2,D}$ are equal when restricted to elements of $\mathcal{H}$ of the type specified by (2.9) and the line following that equation. Combining this commutativity with the unitarity of $\mathcal{U}'_{D,t}$, one sees that the theorem will follow if

$$\operatorname*{s-lim}_{|t|\to\infty}(W_2-W_{2,D})\mathcal{U}'_{D,t}=0. \tag{4.19}$$

In this proof, f will again be fixed and of the form (2.9), with $F\in\mathcal{S}_D$ and with φ a continuous complex-valued function on $\mathbb{R}^{3(N-n)}$ of bounded support. Reasoning similar to that in the proof of Theorem 4.1 shows that (4.19) will hold for the D being considered if we establish that

$$\lim_{|t|\to\infty} \| (W_2 - W_{2,D}) \mathcal{Z}_{D,t} f \| = 0. \tag{4.20}$$

To prove (4.20), we first change the variables of integration in the integral $\| (W_2 - W_{2,D}) \mathcal{Z}_{D,t} f \|^2$ to $X = X_D$, $\xi = \xi_D$ and employ (A3) and the other pertinent definitions, thus obtaining

$$\| (W_2 - W_{2,D}) \mathcal{Z}_{D,t} f \|^2 = \text{const} \, |t|^{-3n} \int_{\mathbb{R}^{3n} \times \mathbb{R}^{3(N-n)}} | \sigma'(X,\xi)\, \lambda(X,\xi,t) \times \hat{F}(t^{-1}\hat{Y}) \varphi(\tilde{\xi}) - \sigma'_D(\xi) \hat{F}(t^{-1}Y) \varphi(\tilde{\xi}_D) |^2 \, d\mathbf{X}\, d\xi, \tag{4.21}$$

where $\sigma'(X,\xi) = \sigma(x)$, $\sigma'_D(\xi) = \sigma_D(x)$ [$\sigma_D(x)$ can be expressed in terms of ξ alone by our assumptions on h_D, including (c)], and the other new symbols in (4.21) have the following meanings. We define

$$\lambda(X,\xi,t) = \exp[\tfrac{1}{2} i t^{-1} \sum_{l=0}^{n} M_l (|\tilde{\mathbf{X}}_l|^2 - |\mathbf{X}_l|^2)] \times \exp\{-i[\boldsymbol{\Gamma}_{D,t}(t^{-1}\tilde{Y}) - \boldsymbol{\Gamma}_{D,t}(t^{-1}Y)]\}, \tag{4.22}$$

with $\boldsymbol{\Gamma}_{D,t}(\cdot)$ given by (A8) of the Appendix. We have set $Y = (M_1\mathbf{X}_1, \ldots, M_n\mathbf{X}_n)$, $x = (\tilde{\mathbf{X}}_1, \ldots, \tilde{\mathbf{X}}_n)$, $\tilde{\mathbf{Y}} = (\tilde{\mathbf{Y}}_1, \ldots, \tilde{\mathbf{Y}}_n)$, $\tilde{\xi} = (\tilde{\boldsymbol{\xi}}_1, \ldots, \tilde{\boldsymbol{\xi}}_{N-n})$, and $\tilde{\xi}_D = (\tilde{\boldsymbol{\xi}}_1^D, \ldots, \tilde{\boldsymbol{\xi}}_{N-n}^D)$. Over the indicated range of indices, $\tilde{\mathbf{X}}_l$ and $\tilde{\boldsymbol{\xi}}_j$ are obtained from $\mathbf{X}_l$ and ξ_j, respectively, by replacing x by $h(x)$, and $\tilde{\boldsymbol{\xi}}_j^D$ is obtained from $\boldsymbol{\xi}_j$ by replacing x by $h_D(x)$ [$\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{N-n}$ were defined in the second paragraph of Sec. 2].

Next we change the variables $\mathbf{X}_l$ of integration in (4.21) to

$$u_l = |t|^{-1} \mathbf{X}_l \quad (l = 1, \ldots, n), \tag{4.23}$$

leaving the remaining N-n variables $\boldsymbol{\xi}_j$ unchanged. In terms of the vector variables $u = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ and $\xi = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{N-n})$ (4.21) assumes the form

$$\| (W_2 - W_{2,D}) \mathcal{Z}_{D,t} f \|^2 = \text{const} \int_{\mathbb{R}^{3n} \times \mathbb{R}^{3(N-n)}} | \sigma''(u,\xi,t) \lambda'(u,\xi,t)\, G(u,\xi,t) - \sigma'_D(\xi)\, G_D(u,\xi,t) |^2 \, du\, d\xi, \tag{4.24}$$

where

$$\sigma''(u,\xi,t) = \sigma'(X,\xi),$$

$$\lambda'(u,\xi,t) = \lambda(X,\xi,t),$$

$$G(u,\xi,t) = \hat{F}(t^{-1}\tilde{Y}) \varphi(\tilde{\xi}),$$

and

$$G_D(u,\xi,t) = \hat{F}(t^{-1}Y)\, \varphi(\tilde{\xi}_D).$$

The following assertions, proved below, entail jointly the truth of (4.20). First, the integrand of (4.24) tends to zero for a.a. (u,ξ) as $|t| \to \infty$. Second, one can interchange the integral in (4.24) with the limit $|t| \to \infty$.

Let $u = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be fixed and such that $\nu(u) = \min_{1 \le l < m \le n} |\mathbf{u}_l - \mathbf{u}_m| > 0$ and also let ξ be fixed. Then

$$\lim_{|t|\to\infty} [\sigma''(u,\xi,t) - \sigma'_D(\xi)] = 0, \tag{4.25a}$$

$$\lim_{|t|\to\infty} [G(u,\xi,t) - G_D(u,\xi,t)] = 0, \tag{4.25b}$$

$$\lim_{|t|\to\infty} \lambda'(u,\xi,t) = 1. \tag{4.25c}$$

The first assertion of the previous paragraph follows by combining (4.25a)–(4.25c) with the boundedness of σ'_D and G_D. This boundedness results from assumption (e) together with the hypotheses that $F \in \mathcal{S}_D$ and that φ is bounded and the appropriate definitions.

Before proving (4.25a)–(4.25c), we mention two essential reasons why these equations hold at the indicated points (u,ξ): (i) h and h_D have been required to agree asymptotically for $\nu(X) \to \infty$ in the sense (4.16); (ii) taking the limit $|t| \to \infty$ in (4.25a)–(4.25c) is roughly the same as letting $\nu(X) \to \infty$.

Equation (4.25a) follows directly at the stated points (u,ξ) by employing the pertinent definitions [including (4.23)] together with (4.16).

To prove (4.25b), we first remark that (b), (c), and (d) entail jointly that the vectors

$$\boldsymbol{\Delta}_l(X,\xi) \equiv \tilde{\mathbf{X}}_l - \mathbf{X}_l = \frac{1}{M_l} \sum_{j \in C_l} m_j \rho_j(X,\xi), \quad l = 1, \ldots, n,$$

and

$$\tilde{\boldsymbol{\xi}}_r - \tilde{\boldsymbol{\xi}}_r^D = \sum_{j=1}^{N-n} b_{rj} [\rho_j(\mathbf{X},\xi) - \rho_j^D(X,\xi)], \quad r = 1, \ldots, N-n,$$

vanish for fixed ξ when $\nu(X) \to \infty$. Here the b_{rj} are constants. Combining this vanishing property with the continuity of $\hat{F}$ and φ and the relevant definitions, we see that (4.25b) is satisfied at the desired (u,ξ) values.

Next we establish (4.25c). The vanishing of $\boldsymbol{\Delta}_l(X,\xi)$ for each l in the last mentioned limit entails directly that the argument of the first exponential in (4.22), when expressed as a function of u, ξ, and t, vanishes when $|t| \to \infty$ for fixed (u,ξ) such that $\nu(u) > 0$.

We proceed to show that the argument of the second exponential in (4.22) has the same property, and hence that (4.25c) is true at each such (u,ξ). Using (A8) of the Appendix, we see that this will follow if for all integers i, j, λ, and μ such that $i \in C_\lambda$, $j \in C_\mu$, $1 \le \lambda < \mu \le n$, the function

$$H_{\lambda i,\mu j}(u,\xi,t) = \int_0^t \left[V_{ij}^L\!\left(\frac{s}{t}(\tilde{\mathbf{X}}_\lambda - \tilde{\mathbf{X}}_\mu)\right) - V_{ij}^L\!\left(\frac{s}{t}(\mathbf{X}_\lambda - \mathbf{X}_\mu)\right) \right] ds$$

vanishes for every given (u,ξ) with $\nu(u) > 0$ as $|t| \to \infty$. Henceforth in this proof, i, j, λ, and μ will be as stated in the previous sentence.

Because of (e), $|\boldsymbol{\Delta}_l(X,\xi)|$ is bounded for each l. Employing this fact, the hypotheses on V_{ij}^L made in the present theorem, an elementary inequality, and the appropriate definitions, one finds for $|s| \le |t| \ne 0$:

$$\left| V_{ij}^L\!\left(\frac{s}{t}(\tilde{\mathbf{X}}_\lambda - \tilde{\mathbf{X}}_\mu)\right) - V_{ij}^L\!\left(\frac{s}{t}(\mathbf{X}_\lambda - \mathbf{X}_\mu)\right) \right| \le \text{const} \left|\frac{s}{t}\right| (|\boldsymbol{\Delta}_\lambda(X,\xi)| + |\boldsymbol{\Delta}_\mu(X,\xi)|) \times [1 + |\boldsymbol{\Delta}_\lambda(X,\xi)| + |\boldsymbol{\Delta}_\mu(X,\xi)|] \left(1 + \left|\frac{s}{t}\right| |\mathbf{X}_\lambda - \mathbf{X}_\mu|\right)^{-(1+\rho)}$$

$$\text{const}\left|\frac{s}{t}\right|(1+|s|\,|\mathbf{u}_\lambda-\mathbf{u}_\mu|)^{-(1+\rho)}, \qquad (4.26)$$

where the proportionality constants can be chosen to be the same for all u, ξ, and t, ρ being the constant greater than $\frac{1}{2}$ mentioned in Theorem 4.1. The estimate in the last line of (4.26) and an elementary computation yield

$$H_{\lambda i,\mu j}(u,\xi,t)=O(|t|^{-\rho})$$

as $|t|\to\infty$ for fixed (u,ξ) such that $\nu(u)>0$. The proof of (4.25c) is complete.

There remains only to justify the validity of interchanging the limit $|t|\to\infty$ with the integral in (4.24). This follows by dominated convergence. Indeed, there exists a positive constant t_2 such that, when $|t|\geq t_2$, (4.24) is dominated for all (u,ξ) by a function proportional to the characteristic function of a compact subset of $\mathbb{R}^{3n}\times\mathbb{R}^{3(N-n)}$, the proportionality constant being independent of t. This dominance is easily proved by using, in particular, the relation $\hat{F}\in C_0^\infty(\mathbb{R}^{3n})$, the boundedness and support properties of φ, and the boundedness of $|\rho_j^D(X,\xi)|$ for every j.

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APPENDIX: PROOF OF AN ASYMPTOTIC TIME-EVOLUTION THEOREM WHEN CERTAIN LONG-RANGE POTENTIALS ARE PRESENT

In this Appendix, we establish an asymptotic theorem on the time evolution of wavepackets under the action of the renormalized propagators $U'_{D,t}$ when m in (3.8) is unity. This theorem applies to types of long-range potentials which include a wide class of Coulombic interactions. Before stating it, some definitions are in order.

Henceforth, we consider a fixed cluster decomposition $D=\{C_1,\ldots,C_n\}$. Since in this Appendix we will only consider the functions $\Gamma_{D,t}^{(r)}(P)$ of (3.3) for $r=1$, we find it convenient to set

$$\Gamma_{D,t}(P)=\Gamma_{D,t}^{(1)}(P) \qquad \text{(A1)}$$

for $P=(\mathbf{P}_1,\ldots,\mathbf{P}_n)\in\mathbb{R}^{3n}$.

Let $\underline{S}_D$ be the set of all complex-valued functions h on $\mathbb{R}^{3n}$ with $L^1(\mathbb{R}^{3n})$ Fourier transforms $\hat{h}\in C_0^\infty(\mathbb{R}^{3n})$ and such that supp h does not intersect the set $\{(\mathbf{P}_1,\ldots,\mathbf{P}_n)\in R^{3n}\,|\,M_k^{-1}\mathbf{P}_k=M_l^{-1}\mathbf{P}_l,\ 1\leq k<l\leq n\}$. It will be convenient to work with elements $f\in H$ of the form.

$$f(x)=F(X_D)\,\varphi(\xi_D), \qquad \text{(A2)}$$

where x, $X_D=(\mathbf{X}_1,\ldots,\mathbf{X}_n)$, and ξ_D are related as in Sec. 2, $F\in\underline{S}_D$, and $\varphi\in L^2(\mathbb{R}^{3(N-n)})(\varphi\equiv 1)$ if $n<N$ $(n=N)$.

When $\Gamma_{D,t}(P)$ exists as a real measurable function, as it does under the assumption of this appendix, there exists a unitary operator $Z_{D,t}: H\to H$, uniquely defined, in particular, by its action

$$(Z_{D,t}f)(x)=\left[\prod_{l=1}^n\left(\frac{M_l}{it}\right)^{3/2}\right]\exp\left(\frac{i}{2t}\sum_{l=1}^n M_l|\mathbf{X}_l|^2\right)\times\exp[-i\Gamma_{D,t}(t^{-1}Y)]\,\hat{F}(t^{-1}Y)\,\varphi(\xi_D) \qquad \text{(A3)}$$

on the elements f of the type (A2) just specified, whose span is dense in H. Here $Y=(M_1\mathbf{X}_1,\ldots,M_n\mathbf{X}_n)$ and $\hat{F}$ may, and will, be interpreted as the $L^1(\mathbb{R}^{3n})$ Fourier transform of F. Hence $\hat{F}$ is in the Schwartz space $S(\mathbb{R}^{3n})$ of functions of fast decrease.

The promised asymptotic result is given by

Lemma A.1: For $1\leq i<j\leq N$, let $V_{ij}^L(.)$ be real functions on $\mathbb{R}^3$ satisfying (3.1) everywhere on $\mathbb{R}^{3N}$ for $p=1$ with a constant $\rho>\frac{1}{2}$. Then

$$\operatorname*{s-lim}_{|t|\to\infty}(U'_{D,t}-Z_{D,t})=0, \qquad \text{(A4)}$$

where $U'_{D,t}$ is the operator (3.8) for $m=1$.

Remarks:

1. When the V_{ij}^L are Coulomb potentials, this lemma follows from an estimate of Dollard.[31]

2. In the case $m=1$, the lemma applies under conditions weaker than the sufficient conditions for the existence of the wave operators in this case stated in Sec. 3. As far as we know, it is an open question whether results of the type (A4) hold when ρ is merely required to be positive and, *a fortiori* under the very weak conditions on the V_{ij}^L considered by Alsholm[32] in the context of single-channel scattering by long-range potentials.

Proof of Lemma A.1: In this proof f will be a fixed element of H as specified by (A2) and by the definitions immediately after that equation.

The existence of the unitary operator $U'_{D,t}(m=1)$ under the hypotheses of the lemma should be clear. Since the linear manifold spanned by elements of the latter type is dense in H, it suffices to show that

$$\lim_{|t|\to\infty}\|(U'_{D,t}-Z_{D,t})f\|=0 \qquad \text{(A5)}$$

for the operators under discussion, $\|\cdot\|$ again denoting the $L^2(\mathbb{R}^{3N})$ norm.

In the case $m=1$ of interest, the relevant definitions of this Appendix and of Sec. 3 and standard properties of Fourier transforms yield

$$[(U'_{D,t}-Z_{D,t})f](x)=\left[\prod_{l=1}^n\left(\frac{M_l}{it}\right)^{3/2}\right]\exp\left(\frac{i}{2t}\sum_{l=1}^n M_l|\mathbf{X}_l|^2\right)\times\hat{G}_t(t^{-1}Y)\,\varphi(\xi_D),$$

where $\hat{G}_t$ is the $L^2(\mathbb{R}^{3n})$ Fourier transform of

$$G_t(Z)=\left\{\exp\left[\frac{i}{2t}\sum_{l=1}^n M_l|Z_l|^2\right]-1\right\}F_t(Z)$$

for $Z=(Z_1,\ldots,Z_n)\in\mathbb{R}^{3n}$. Here F_t is such that its $L^2(\mathbb{R}^{3n})$ Fourier-transform is given by

$$\hat{F}_t(Z)=\exp[-i\Gamma_{D,t}(Z)]\hat{F}(Z).$$

Directly from the definitions, making use of the assumed properties of the pertinent functions and proceeding in a manner similar to that of Reed and Simon,[33] we obtain for $t \neq 0$:

$$
\begin{aligned}
&\| (U'_{D,t} - Z_{D,t}) f \| \\
&= \left[\prod_{l=1}^{n} \left(\frac{M_l}{|t|} \right)^{3/2} \right] \left\| \hat{G}_t \left(\frac{M_1}{t} \cdot , \ldots , \frac{M_n}{t} \cdot \right) \right\|_2 \\
&= \| \tilde{G}_t(\cdot , \ldots , \cdot) \|_2 = \| G_t(\cdot , \ldots , \cdot) \|_2 \\
&\leq \left(\frac{2}{|t|} \right)^{1/2} \left\| \left(\sum_{l=1}^{n} M_l \left| \mathbf{Z}_l \right|^2 \right)^{1/2} F_t \right\|_2 \\
&\leq \left(\frac{2}{|t|} \right)^{1/2} \left\| \sum_{l=1}^{n} M_l^{1/2} \left| \mathbf{Z}_l \right| F_t \right\|_2 \\
&\leq \left(\frac{2}{|t|} \right)^{1/2} \sum_{l=1}^{n} M_l^{1/2} \left\| \left| \mathbf{Z}_l \right| F_t \right\|_2 \\
&= \left(\frac{2}{|t|} \right)^{1/2} \sum_{l=1}^{n} M_l^{1/2} \left\| \left| \nabla_{P_l} \hat{F}_t \right| \right\|_2 \qquad \text{(A6)} \\
&\leq \left(\frac{2}{|t|} \right)^{1/2} \sum_{l=1}^{n} M_l^{1/2} \left[\| \hat{F} | \nabla_{P_l} \mathbf{\Gamma}_{D,t} | \|_2 + \| | \nabla_{P_l} \hat{F} | \|_2 \right],
\end{aligned}
$$

where $\| \cdot \|_2 = \| \cdot \|_{L^2(\mathbb{R}^{3n})}$ and where the notation $|z| = (\sum_{i=1}^{3} \bar{z}_i z_i)^{1/2}$ has been used for $z = (z_1, z_2, z_3) \in \mathbb{C}^3$. In (A6), we have employed the inequality

$$|\exp(iy) - 1| \leq 2 |y|^{1/2}, \quad y \in \mathbb{R},$$

Minkowski's inequality, and familiar properties of Fourier transforms. Specifically, in obtaining the penultimate from the antepenultimate line of (A6), we have used the facts that $F_t \in L^2(\mathbb{R}^{3n})$, that $\hat{F}_t$ is differentiable, that $\hat{F}_t$ and $\Delta_{P_l} \hat{F}_t$ are in $L^1(\mathbb{R}^{3n}) \cap L^2(\mathbb{R}^{3n})$ $(l = 1, \ldots , n)$ and that $\hat{F}(P)$ tends to zero as the $\mathbb{R}^{3n}$ norm of P tends to infinity.

Let $P = \{\mathbf{P}_1, \ldots , \mathbf{P}_n\} \in \mathbb{R}^{3n}$ be such that for each $1 \leq k < l \leq n$, $M_k^{-1}\mathbf{P}_k - M_l^{-1}\mathbf{P}_l$ lies in a fixed compact subset of $\mathbb{R}^3 - \{0\}$. Then

$$|\nabla_{P_k} \Gamma_{D,t}(P)| = O(|t|^{1-\rho}), \quad k = 1, \ldots , n, \qquad \text{(A7)}$$

for $|t| \to \infty$ if the hypotheses of the present lemma are satisfied and if, in addition, $\rho < 1$. The latter condition entails no loss of generality. The constant entailed by the O symbol in (A7) depends on the above compact subset. One can derive (A7) by an elementary estimate of the gradients ∇_{P_l} of the rhs of the equation

$$\mathbf{\Gamma}_{D,t}(P) = \sum_{1 \leq \lambda < \mu \leq n} \sum_{\substack{i \in C_\lambda \\ j \in C_\mu}} \int_0^t V_{ij}^L (s(M_\lambda^{-1}\mathbf{P}_\lambda - M_\mu^{-1}\mathbf{P}_\mu))\, ds, \qquad \text{(A8)}$$

which is itself an immediate consequence of (A1), (3.3), and (3.4).[34]

Employing, in particular, (A6), (A7), and the assumption $F \in \mathcal{S}_D$ we directly conclude that

$$\|(U'_{D,t} - Z_{D,t}) f \| = O(|t|^{1/2-\rho}) \to 0$$

as $t \to \infty$ if $\frac{1}{2} < \rho < 1$. Hence (A5) obtains under the conditions stated in the lemma.

[1]A. W. Sáenz and W. W. Zachary, J. Math. Phys. **17**, 409 (1976). [Erratum: J. Math. Phys. **17**, 1616 (1976). In addition, insert "on $C_0^\infty(\mathbb{R}^3)$" after $-\Delta + V$, p. 410, left-hand column, line 5; replace "sum of the quadratic forms of H_0 and V" by "in the quadratic form sense," p. 410, left-hand column, lines 7 and 8; replace s by $2s$ in the last displayed equation of p. 417; and replace "in the closure of compact operators with respect to H_0-convergence" by "in $K^+ \cap K^-$, where $K^\pm$ is the closure of the compact operators on $L^2(\mathbb{R}^3)$ with respect to $\pm H_0$-convergence," p. 418, right-hand column, lines 2 and 3 of Ref. 7.]

[2]H. Ekstein, Phys. Rev. **117**, 1590 (1960).

[3]A. W. Sáenz and W. W. Zachary, Phys. Lett. B **58**, 13 (1975). [Delete the sentence following Eq. (4) of this reference.]

[4]M. I. Haftel, Phys. Rev. C **14**, 698 (1976).

[5]M. I. Haftel and W. M. Kloet, Phys. Rev. C **15**, 404 (1977).

[6]C. Chandler and A. G. Gibson, J. Math. Phys. **14**, 1328 (1973).

[7]Work of this kind was pioneered by F. Coester, S. Cohen, B. Day, and C. M. Vincent, Phys. Rev. C **1**, 769 (1970). Further papers applying unitary operators of the type introduced in this reference to nuclear physics questions are cited in Refs. 1, 3, 4, and 5.

[8]Suppose that H is expressed formally by (2.1) and that each $V_{ij}(\cdot)$ in that equation is an appropriate element of $L^2_{\text{loc}}(\cdot)$. As is well known, one can then define for each $D = \{C_1, \ldots , C_n\}$ and $l = 1, \ldots , n$ with $n_l \geq 2$ the operators H, H_D, and h_l as the unique self-adjoint operators in $L^2(\mathbb{R}^{3N})$, $L^2(\mathbb{R}^{3N})$, and $L^2(\mathbb{R}^{3(n_l-1)})$, respectively, which are extensions of the pertinent differential operators acting on $C_0^\infty(\mathbb{R}^{3N})$, $C_0^\infty(\mathbb{R}^{3N})$, and $C_0^\infty(\mathbb{R}^{3(n_l-1)})$, respectively. See, e.g., K. Jörgens and J. Weidmann, *Spectral Properties of Hamiltonian Operators* [Lecture Notes in Mathematics, Vol. 313, (Springer-Verlag, Berlin, 1973)]. Another typical case is when the V_{ij} are mixtures of nonsingular and singular potentials as specified in pp. 3, 4 of Ref. 13, in which case all these Hamiltonians exist as Friedrichs extensions. In either of these two cases, H_D obeys (2.2) for all D. Multiparticle Hamiltonians, defined in the quadratic form sense, have been discussed by B. Simon, *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms* (Princeton U. P., Princeton, N.J., 1971).

[9]W. Hunziker, J. Math. Phys. **6**, 6 (1965).

[10]Henceforth, all statements and equations involving operators with the subscript t should be understood to hold for each $-\infty < t < \infty$.

[11]For such systems, the existence of the wave operators $\Omega_D^\pm = \text{s-lim}_{t \to \pm\infty} \exp(itH) \exp(-itH_D)$ for all D, and hence of the $\Omega_\alpha^\pm$ of type (2.5) for all channels, was first proved by Hack.[12] His results were generalized by Hunziker,[13] who showed that all the $\Omega_D^\pm$ exist when the V_{ij} are suitable mixtures of singular and nonsingular short-range potentials, a result which he later[14] generalized to allow for the presence of hard cores.

[12]M. N. Hack, Nuovo Cimento **13**, 231 (1959).

[13]W. Hunziker, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. XA.

[14]W. Hunziker, Helv. Phys. Acta **40**, 1052 (1967).

[15]This property was first proved for $N = 3$ by Faddeev[16] and later by others.[17] Hepp[18] gave a partial proof for $N \geq 2$ and proofs for $N \geq 2$ when only the free channel exists are known.[19] Without this restriction, asymptotic completeness has been proved recently[20] for $N = 4$ and a time-dependent version of this attribute has been established by Sigal[21] for $N \geq 2$.

[16]L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Israel Program for Scientific Translations, Jerusalem, 1965).

[17]J. Ginibre and M. Moulin, Ann. Inst. Henri Poincaré **A21**, 97 (1974); L. E. Thomas, Ann. Phys. (N.Y.) **90**, 127 (1975); E. Mourre, Ann. Inst. Henri Poincaré **A26**, 219 (1977).

[18]K. Hepp, Helv, Phys. Acta **42**, 425 (1969).

[19]R. J. Iorio and M. O'Carroll, Commun. Math. Phys. **27**, 137 (1972); P. Ferrero, O. de Pazzis, and D. W. Robinson, Ann. Inst. Henri Poincaré **A21**, 217 (1974).

[20]G. A. Hagedorn, Bull. Am. Math. Soc. **84**, 155 (1978); Thesis, Princeton Univ., 1978.

[21]I. M. Sigal, "Mathematical Foundations of Scattering Theory for Multiparticle Systems," Tel-Aviv University preprint (1975); Bull. Am. Math. Soc. **84**, 152 (1978) and Mem. Am.

Math. Soc. (in press).

[22]Here is an example of a unitary operator for which (2.7) does not hold for all D. Let H be a self-adjoint extension of a differential operator of the form (2.1) for $N \geq 2$, with all the V_{ij} zero except for V_{12}, which is an operator of multiplication in $L^2(\mathbb{R}^{3N})$ by a real-valued function $v(|\mathbf{x}_1 - \mathbf{x}_2|)$, say in $L^2_{\rm loc}(\mathbb{R}^3)$. This function is assumed to be such that the free-channel Møller wave operators (2.5), say $\Omega_0^{\pm}$, exist, are unitary, and have the property $\Omega_0^+ \neq \Omega_0^-$, so that $S_{00} \equiv \Omega_0^{+*}\Omega_0^- \neq I$. Since in this example $\Omega_0^{\pm}$ are related in an obvious manner to the wave operators for scattering of particles 1 and 2 in the center-of-mass frame, it is known that many such v exist. Setting $W = \Omega_0^-$, one readily proves for the free channel that the rhs of (2.7) is S_{00} for $t \to \infty$ and I for $t \to -\infty$. This example is a variant of one devised by Ekstein[2] (See also p. 411 of Ref. 1).

[23]Satisfactory necessary and sufficient conditions for the solubility of an analogous, simpler problem arising in the single-channel context of Ref. 1 are known. In a more general setting than that of Ref. 1, this problem is to determine all unitary operators $U:\mathcal{H}' \to \mathcal{H}'$ on a separable Hilbert space $\mathcal{H}'$ such that $\mu_+(U) = \mu_-(U) =$ unitary operator. Here $\mu_{\pm}(B) = \text{s-lim}_{t\to\pm\infty} \exp(itA)B\exp(-itA)$ for each $B:\mathcal{H}' \to \mathcal{H}'$ for which the respective limits exist, A being a *fixed* self-adjoint operator in $\mathcal{H}'$ whose spectrum is absolutely continuous [in Ref. 1, $U = W$, $A = H_0$, and $\mu_{\pm}(U) = W_{\pm}$]. It is elementary to prove that such unitary operators U are those of the form $U_0 + Z$, where $U_0:\mathcal{H}' \to \mathcal{H}'$ is unitary and commutes with A and $Z \in \text{Ker}\mu_+ \cap \text{Ker}\mu_-$. The nontrivial problem of characterizing the kernels $\text{Ker}\mu_{\pm}$ of $\mu_{\pm}$ was solved by H. Baumgärtel, Math. Nachr. **58**, 279 (1973), Theorem 1, for more general $\mu_{\pm}$ than those considered here.

[24]The statement is that $\text{s-lim}_{\nu(a)\to\infty} T^*_{D,a} W T_{D,a}$ exists and equals both limits $\text{s-lim}_{t\to\pm\infty} \mathcal{U}^*_{D,t} W \mathcal{U}_{D,t}$ for the $D = \{C_1, \ldots, C_n\}$ considered. Here $T_{D,a}$, where $a = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \in R^{3n}$, translates each center of mass $\mathbf{X}$ by $\mathbf{a}_l$: $(T_{D,a}g)(x) = G(X_D + a, \xi_D)$, for each $g \in \mathcal{H}$, where $G(X_D, \xi_D)$ is $g(x)$ expressed in terms of center-of-mass and internal variables. We define $\nu(a) = \min_{1\leq i<j\leq n} |\mathbf{a}_i - \mathbf{a}_j|$. The penultimate sentence of Ref. 22 entails that this equality of the strong limits $\nu(a) \to \infty$ and $t \to \pm\infty$ does not hold for the example $W = \Omega_0^-$ of that Reference. [A more general result of Hunziker[9] implies that $\text{s-lim}_{\nu(a)\to\infty} \times T^*_{D,a} W T_{D,a} = I$ for that example if $v(|\mathbf{x}|)$ is square-integrable over R^3.] The condition that these three strong limits exist and are equal for each D is expected to be satisfied by physically reasonable operators W and obtains for the pertinent examples of Sec. 4.

[25]Theorem 1 of Ref. 3 can be proved similarly.

[26]Unfortunately, the proof of the existence of multichannel wave operators for long-range potentials given by one of us (W.W.Z.) in J. Math. Phys. **17**, 1056 (1976) is incorrect except for a relatively special case [Erratum: J. Math. Phys. **18**, 536 (1977)]. The generalized existence proof to which we allude in the text establishes the existence of the wave operators $\text{s-lim}_{t\to\pm\infty} \exp(itH)\exp(-itH_D)\exp(-iG^{(m)}_{D,t})$ for each D by (essentially) reducing the problem to the corresponding one for the free channel. Under the same conditions imposed in this proof, these wave operators have the usual intertwining property and the ranges of the wave operators $\Omega^{\pm}_{\alpha}$ of type (3.6) are pairwise orthogonal.

[27]More precisely, for physically reasonable operators W, one expects that $\text{s-lim}_{\nu(a)\to\infty} T^*_{D,a} W T_{D,a}$ exists and equals both limits $\text{s-lim}_{t\to\pm\infty} \mathcal{U}^*_{D,t} W \mathcal{U}_{D,t}$ for each D. There are unitary operators W for which this equality fails, namely $W = S_{00}$, where S_{00} is the S operator in Ref. 22, with $v(r)$ a repulsive Coulomb potential. [This can be established by using, in particular, a result of W. Ross, quoted as Eq. (IV.10) by I.W. Herbst, Commun. Math. Phys. **35**, 193 (1974).] The stated equality of these spatial and temporal limits obtains for each D for the examples of operators W in Sec. 4.

[28]There is extensive literature on the use of hyperspherical coordinates in nuclear physics. See, e.g., the following references and the pertinent papers cited therein: M. Fabre de la Ripelle, C.R. Acad. Sci. Paris **268**, B121 (1969); **269**, B1070 (1969); Sov. J. Nucl. Phys. **13**, 279 (1971); and A.M. Badalyan, E.S. Gal'pern, V.N. Lyakhovitskiĭ, V.V. Pustovalov, Yu. M. Simonov, and E.L. Surkov, Sov. J. Nucl. Phys. **6**, 345 (1968).

[29]Let $W_0:\mathcal{H} \to \mathcal{H}$ be unitary and of the form $I + K_0$, where K_0 is compact. Then $\text{s-lim}_{t\to\pm\infty} \mathcal{U}^*_{D,t} W_0 \mathcal{U}_{D,t} = I$ for all D and therefore properties (1) and (2) of Sec. 2 hold for each D with $W = W_0$. Indeed, for every D one has $\text{w-lim}_{t\to\pm\infty} \mathcal{U}_{D,t} = 0$ and hence $\text{s-lim}_{t\to\pm\infty} K\mathcal{U}_{D,t} = 0$. But this example is physically unnatural. In fact, suppose that the wave operators $\Omega^{\pm}_{\alpha}$ of type (2.5) corresponding to an N-particle system with Hamiltonian H obey an equation of type (4.1) for each channel α and each of the stated functions g, with W and w in (4.1) replaced by $\Omega^{\pm}_{\alpha}$ and by the pertinent wave operators $\omega^{\pm}_{\alpha}$ in the center-of-mass system, respectively. Now, the compactness of K_0 makes it impossible for W_0 to obey (4.1) for all such g. Hence, the wave operators $\widetilde{\Omega}^{\pm}_{\alpha} = W^*_0 \Omega^{\pm}_{\alpha}$ pertaining to the N-particle system with Hamiltonian $W^*_0 H W_0$ do not have for any α the tensor product structure (4.1) possessed by the operators $\Omega^{\pm}_{\alpha}$ just mentioned. In other words, W_0 fails to preserve the independence of the center-of-mass and relative-motion degrees of freedom in scattering events which holds for the system with Hamiltonian H. The fact that W_0 does not possess the indicated tensor-product structure makes it useless for the practical nuclear physics applications discussed in Refs. 3–5.

[30]D. Bohn, E.P. Gross, and G.A. Baker pioneered the use of such operators in the context of nonrelativistic two-body quantum mechanics in the center-of-mass frame [See Ref. 21 of Ref. 1]. M. Eger and E.P. Gross, Ann. Phys. (N.Y.) **24**, 63 (1963) defined N-particle transformations of the type (4.17) and applied them to many-body physics, but did not consider the question of scattering equivalence.

[31]J.D. Dollard, thesis, Princeton University, 1963, estimate (52), p. 126.

[32]P.K. Alsholm, thesis, University of California, Berkeley, 1972.

[33]M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1975), Vol. II, pp. 60–1.

[34]The estimate (A7) is a very special case of estimates obtainable for the functions $\Gamma^{(r)}_{D,t}(P)$ $(r \geq 1)$, defined in (3.3) by the use of the methods of Ref. 32.

Ground state representation of the infinite one-dimensional Heisenberg ferromagnet. III. Scattering theory

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This article gives a complete description of the scattering for the spin 1/2 Heisenberg ferromagnetic chain in its ground state representation.

1. INTRODUCTION

The purpose of this article is to give a complete description of the scattering for the spin-$\frac{1}{2}$ one-dimensional Heisenberg chain with nearest neighbor interactions, in its ground state representation.[1–4] In Ref. 4, an explicit, complete eigenfunction expansion based on Bethe's solution[1] was obtained for the ground state Hamiltonian. Here, we use this eigenfunction expansion to obtain explicit expressions for the wave operators and S matrix.

The qualitative picture for the scattering is a simple one. Recall that the ground state Hamiltonian commutes with a spin wavenumber operator. The Hamiltonian, restricted to its N-spin wave sector, is unitarily equivalent in a natural way to a second differencelike operator $-\Delta_N$ acting in an l^2-space. Within the N-spin wave subspace, the N spin waves can combine to form bound state complexes. The manner in which they combine, e.g., n_1 unbound spin waves, n_2 two-spin wave complexes, etc., with $\sum_j jn_j = N$ we refer to as an N-binding. The scattering therefore involves channels. However, we show that no inelastic processes occur, i.e., the binding is preserved in a scattering process, a result which seems to have been known already for the anisotropic ferromagnetic chain[5] and the one-dimensional N-body problem with repulsive or attractive δ-function interaction.[6] The S matrix, restricted to a particular N-binding in the appropriate (momentum) representation, is thus multiplication by a phase function of modulus one, which we compute explicitly for each N-binding.

The form of this phase can be described pictorially as follows: At $t=0$, imagine r nonoverlapping wavepackets on a line with sharply peaked velocities. As $t\to\pm\infty$, and depending on the relative velocities of the packets, some of the packets will necessarily penetrate each other. The corresponding wave operator will then be a product of phase factors, one for each penetration. Note that for unequal velocities, two wavepackets penetrate each other, either for $t\to+\infty$ or for $t\to-\infty$. The S matrix itself will also be a product of such factors. For the spin wave model the velocities of the complex packets are in fact their group velocities.

In Sec. 2 we review the ground state representation, set up notation and define the wave operators and S matrix. The wave operators and S matrix are calculated in Sec. III. An appendix is included in which a particular limit needed in Sec. III is computed.

2. NOTATION AND DEFINITION OF THE WAVE OPERATORS

Wherever possible we follow the notation of Refs. 3, 4. Let $\hat{\mathbf{Z}}^N=\{m=(m_1, m_2, \ldots, m_N)\in \mathbf{Z}^N \mid m_1<m_2<\cdots m_N\}$. Then the generalized eigenfunctions of $-\Delta_N$ acting in $l^2(\hat{\mathbf{Z}}^N)$ are described as follows. Let $\beta=(n_1, n_2, \ldots, n_N)$ with $n_j\geq 0$ and $\sum_j^N jn_j=N$ be an N-binding; n_j is the number of j-spin wave bound state complexes. Partition $\{1,\ldots,N\}$ into a disjoint set of intervals $I_{jk}=\{N_{jk}+1,\ldots,N_{jk}+j\}$ with $N_{jk}=\sum_{l=1}^{j-1} ln_l+(k-1)j$ for $k=1,\ldots,n_j$, $j=1,\ldots,N$. Let $\mathbf{S}_N$ be the permutation group of $\{1,\ldots,N\}$ and let $P_\beta=\{P\in\mathbf{S}_N \mid P(N_{jk}+1)<P(N_{jk}+2)<\cdots<P(N_{jk}+j)$ for each $jk\}$. Set $\mathbf{z}\equiv(z_1, z_2, \ldots, z_N)\in\mathbf{C}^N$, $\mathbf{z}_\beta\equiv(z_{11}, z_{12},\ldots,z_{1n_1}, z_{21},\ldots,z_{Nn_N})$, $z_{jk}\in\mathbf{C}$, (the variables z_{jk} are suppressed if $n_j=0$), and

$$z^{mP}\equiv z_1{}^{m_{P(1)}}z_2{}^{m_{P(2)}}\cdots z_N{}^{m_{P(N)}}.$$

Let $\Gamma_j=\{z\in\mathbf{C} \mid |jz-j+1|=1\}$, $\hat{\Gamma}_\beta=\{\mathbf{z}_\beta \mid z_{jk}\in\Gamma_j$, $0\leq\arg(jz_{jk}-j+1)\leq\arg(jz_{jk'}-j+1)\leq 2\pi$ if $k<k'\}$. The variable z_{jk} parametrizes the momentum of the jkth complex $1\leq k\leq n_j$, i.e., the kth complex consisting of j bound spin waves. Define the fractional linear transformation

$$t^l(z)=\frac{(l+1)z-l}{lz-l+1},\quad z\in\mathbf{C},\quad l\in\mathbf{Z} \tag{2.1}$$

and

$$\exp(-i\varphi_P)(z)=\prod_{\substack{i<j \text{ with}\\ P(i)>P(j)}}\left(-\frac{z_iz_j-2z_j+1}{z_iz_j-2z_i+1}\right) \tag{2.2}$$

for $N\geq 2$. For $\mathbf{z}_\beta$, $\psi_\beta(\mathbf{z}_\beta, \mathbf{m})$ is a generalized eigenfunction for $-\Delta_N$, with

$$\psi_\beta(\mathbf{z}_\beta,\mathbf{m})\equiv\sum_{P\in P_\beta}\mathbf{z}^{\mathbf{m}P}\exp(-i\varphi_P),\quad \mathbf{z}_\beta\in\hat{\Gamma}_\beta \tag{2.3}$$

and it is understood that $z_{N_{jk}+j-l}=t^l(z_{jk})$ if $0\leq l<j$.

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The corresponding eigenvalue is given by

$$\epsilon_\beta(\mathbf{z}_\beta)=-\sum_j^N\sum_{k=1}^{n_j}\frac{j(z_{jk}-1)^2}{2(jz_{jk}-j+1)},\quad \mathbf{z}_\beta\in\hat{\Gamma}_\beta. \tag{2.4}$$

{Equivalently, in a notation closer to that of Bethe,[1]

$$\epsilon_\beta=\sum_l^N[1-\tfrac{1}{2}(z_l+z_l^{-1})] \tag{2.4'}$$

with $z_l=e^{ik_l}$. In general, the k_l's are complex. } The associated Plancherel measure $\mu_\beta(\mathbf{z}_\beta)d\mathbf{z}_\beta$ on $\hat{\Gamma}_\beta$ is given by

$$\mu_\beta(\mathbf{z}_\beta)=\prod_j^N\prod_{k=1}^{n_j}\left[\frac{(-1)^{j-1}}{2\pi i}\;[(j-1)!]^2\frac{j}{jz_{jk}-j+1}\times\prod_{l=1}^{j-1}\left(\frac{z_{jk}-1}{lz_{jk}-l+1}\right)^2\right]. \tag{2.5}$$

The ψ_β's, in association with these measures, provide a complete orthogonal eigenfunction expansion for $-\Delta_N$.[3,4]

Let U_β: $l^2(\hat{\mathbf{Z}}^N)\to L^2(\hat{\Gamma}_\beta,\mu_\beta)$ be defined by

$$f\to\langle\psi_\beta(\mathbf{z}_\beta),f\rangle_{l^2(\mathbf{z}^N)}$$

and let E_β be the projection onto the range of U_β^{-1}. (The E_β's are mutually orthogonal for distinct β's; they are complete in the sense that their sum is the identity.) Set $E_j=E_{(0,0,\ldots,1)}$, where $(0,0,\ldots,1)$ is a j-tuple.

Define the β-channel Hilbert space $\mathcal{H}_\beta^0$ to be the range of $U_\beta^{0^{-1}}$: $L^2(\hat{\Gamma}_\beta,\mu_\beta)\to\otimes_j\otimes^{n_j}E_jl^2(\hat{\mathbf{z}}^j)$ defined by $f\to\int\psi_\beta^0(\mathbf{z}_\beta)f(\mathbf{z}_\beta)\mu_\beta(\mathbf{z}_\beta)d\mathbf{z}_\beta$ with $\psi_\beta^0=\otimes_j\otimes^{n_j}\psi_j(z_{jk})$ and $\psi_j(z_{jk})=\psi_{(0,0,\ldots,1)}(z_{jk})$, $(0,0,\ldots,1)$ a j-tuple. Thus U_β^0: $\mathcal{H}_\beta^0\to L^2(\hat{\Gamma}_\beta,\mu_\beta)$ is unitary. Define the channel Hamiltonian $-\Delta_\beta^0$ acting in $\mathcal{H}_\beta^0$ by

$$-\Delta_\beta^0=-\sum_j\sum_{k=1}^{n_j}\mathbf{1}^{(\Sigma_{j'=1}^{j-1}n_{j'})}\otimes\mathbf{1}^{(k-1)}\otimes\Delta_j\otimes\mathbf{1}^{(n_j-k)}\otimes\mathbf{1}^{\Sigma_{j'=j+1}^{N}n_{j'}}. \tag{2.6}$$

Let $J_\beta:\mathcal{H}_\beta^0\to l^2(\hat{\mathbf{Z}}^N)$ be defined by

$$J_\beta f(m_1,m_2,\ldots,m_N)=\sum_{P\in p_\beta}f(m_{P(1)},m_{P(2)},\ldots,m_{P(N)}). \tag{2.7}$$

Finally, define the channel wave operators $W_\pm(\beta):\mathcal{H}_\beta^0\to L^2(\hat{\mathbf{Z}}^N)$

$$W_\pm(\beta)=\operatorname*{s-lim}_{t\to\pm\infty}\exp(-it\Delta_N)J_\beta\exp(it\Delta_\beta^0) \tag{2.8}$$

and the S matrix

$$S_{\alpha\beta}=W_+^*(\alpha)W_-(\beta). \tag{2.9}$$

Remark; Our motivation for the definition (2.8) is the analogy with Schrödinger operators in one dimension with Bose statistics. For example, suppose in the Schrödinger case an N-body state evolves asymptotically into a β-channel state,

$$\exp(-itH)\psi(x_1,\ldots,x_N)\to(N!)^{-1/2}\sum_P\mathbf{P}\exp(-itH_\beta^0)\times\psi_\pm(x_1,\ldots,x_N)$$

with P a permutation operator. Note that the rhs of this expression must be symmetric, hence the symmetrizing operation. The factor $(N!)^{-1/2}$ is the correct one since the individual terms in the sum are asymptotically mutually orthogonal. This suggests defining the wave operators $W_\pm(\beta)=(N!)^{-1/2}\text{s-lim}\,e^{+itH}\sum_P P\times\exp(-itH_\beta^0)$. If we regard the wave operators as acting into $L^2(\hat{\mathbf{R}}^N)$ ($\hat{\mathbf{R}}^N=\{\mathbf{x}\in\mathbf{R}^N\,|\,x_1\le\cdots\le x_N\}$) instead of $L^2(\mathbf{R}^N)$, we obtain

$$W_\pm(\beta)=\text{s-lim}\exp(itH)\textstyle\sum_P P\exp(-itH_\beta^0),$$

the analog of (2.8).

3. THE SCATTERING MATRIX AND WAVE OPERATORS

Recall the definition of the basic phase factors $\exp(-i\varphi_{ij})\equiv(z_iz_j-2z_j+1)/(z_iz_j-2z_i+1)$ (Ref. 4, Sec. 2). In terms of these factors, we define the phase factor

$$\exp(-i\varphi_{jk,j'k'})(\mathbf{z}_\beta)$$

$$\equiv\prod_{\substack{1\le m\le j\\ 1\le m'\le j'}}\exp(-i\varphi_{N_{jk}+m,N_{j'k'}+m'})$$

$$\equiv\prod_{\substack{0\le l<j\\ 0\le l'<j'}}\left(-\frac{t^l(z_{jk})t^{l'}(z_{j'k'})-2t^{l'}(z_{j'k'})+1}{t^l(z_{jk})t^{l'}(z_{j'k'})-2t^l(z_{jk})+1}\right), \tag{3.1}$$

where $z_{N_{jk}+m}=t^{j-m}(z_{jk})$ for $1\le m\le j$. Since $\overline{t^l(z_{jk})}=[t^{j-l-1}(z_{jk})]^{-1}$ for $\mathbf{z}_\beta\in\hat{\Gamma}_\beta$, it follows that

$$\overline{\exp(-i\varphi_{N_{jk}+m,N_{j'k'}+m'})}=\exp(i\varphi_{N_{jk}+j-m+1,N_{j'k'}+j'-m'+1})$$

and hence that $\exp(-i\varphi_{jk,j'k'})$ indeed has unit modulus. We write $jk<j'k'$ if $j<j'$ or $j=j'$ but $k<k'$. The main results are the following theorem and corollary.

Theorem 1: The wave operators satisfy $W_\pm(\beta)=E_\beta W_\pm(\beta)$. The operators $\tilde{W}_\pm(\beta)\equiv U_\beta W_\pm(\beta)U_\beta^{0^{-1}}$: $L^2(\hat{\Gamma}_\beta,\mu_\beta)\to L^2(\hat{\Gamma}_\beta,\mu_\beta)$ are multiplication operators with multiplication functions given a.e. respectively by

$$\prod_{\substack{jk<j'k'\\ \nu_{jk}>\nu_{j'k'}}}\exp(i\varphi_{jk,j'k'}),\qquad\prod_{\substack{jk<j'k'\\ \nu_{jk}<\nu_{j'k'}}}\exp(i\varphi_{jk,j'k'}),$$

with

$$\nu_{jk}\equiv i\left(\frac{jz_{jk}-j+1}{j}\right)\frac{\partial\epsilon_\beta}{\partial z_{jk}}. \tag{3.2}$$

Remark: The quantity ν_{jk} is the *group velocity* of the jkth complex, i.e., the derivative of the energy with respect to the momentum κ_{jk} with $\exp(i\kappa_{jk})\equiv jz_{jk}-j+1$. Thus the wave operators are simply products of phase factors, one factor for each pair of complexes which penetrate each other in the evolution $t\to\infty$ or $t\to-\infty$.

From Theorem 1, the mutual orthogonality of the projections E_β[4] and the definition of the S matrix, we have the corollary:

Corollary 1: The S matrix satisfies $S_{\alpha\beta}=0$ for $\alpha\neq\beta$. For $\alpha=\beta\,\tilde{S}_{\beta\beta}\equiv U_\beta^0 S_{\beta\beta}U_\beta^{0^{-1}}: L^2(\hat{\Gamma}_\beta,\mu_\beta)\to L^2(\hat{\Gamma}_\beta,\mu_\beta)$ is multiplication a.e. by

$$\prod_{\substack{jk<j'k'\\ \nu_{jk}>\nu_{j'k'}}}\exp(-i\varphi_{jk,j'k'})\prod_{\substack{jk<j'k'\\ \nu_{jk}<\nu_{j'k'}}}\exp(i\varphi_{jk,j'k'}).$$

Proof of Theorem 1: The proof that $W_\pm(\beta)=E_\beta W_\pm(\beta)$ is essentially the same as that used to establish orthogonality of the eigenfunction expansion in Ref: 4; see particularly Lemmas 3.1.3, 3.1.4. We need to show that, for $\alpha\neq\beta$, $E_\alpha W_\pm(\beta)=0$. Let $f\in L^2(\hat{\Gamma}_\alpha,\mu_\alpha)$, $g\in L^2(\hat{\Gamma}_\beta,\mu_\beta)$ be C^∞ functions with f having support away from the analytic sets of codimension one in $\hat{\Gamma}_\alpha$ corresponding to the singularities of the phase factors $\exp(-i\varphi_{ij})$ in ψ_α, and, in addition, both f, g having support away from the hypersurfaces $z_{jk}=1$ in $\hat{\Gamma}_\alpha$, $\hat{\Gamma}_\beta$, respectively, for each jk. Such f's and g's are dense in their respective Hilbert spaces, and it suffices to show that

$$\langle U_\alpha^{-1}f,\ W_\pm(\beta)U_\beta^{0^{-1}}g\rangle\equiv\lim_{t\to\pm\infty}A_t^0(f,g)=0, \tag{3.3}$$

where

$$A_t^0(f,g)\equiv\sum_{\mathbf{m}\in\mathbf{Z}^N}\int\bar{f}(\mathbf{z}'_\alpha)\bar{\psi}_\alpha(\mathbf{z}'_\alpha,\mathbf{m})J\psi_\beta^0(\mathbf{z}_\beta,\mathbf{m})g(\mathbf{z}_\beta)$$
$$\times\exp\{it[\epsilon_\alpha(\mathbf{z}'_\alpha)-\epsilon_\beta(\mathbf{z}_\beta)]\}\mu_\alpha(\mathbf{z}'_\alpha)\mu_\beta(\mathbf{z}_\beta)d\mathbf{z}'_\alpha d\mathbf{z}_\beta. \tag{3.4}$$

Now

$$J\psi_\beta^0=\sum_{P\in P_\beta}\mathbf{z}^{\mathbf{m}_P}$$

again with

$$z_{N_{jk}+j-l}=t^l(z_{jk}),\quad 0\leq l<j,$$

and as a distribution,

$$\sum_{\mathbf{m}\in\mathbf{Z}^N}\bar{\psi}_\alpha(\mathbf{z}'_\alpha,\mathbf{m})\psi_\beta(\mathbf{z}_\beta,\mathbf{m})$$
$$=2\pi\sum_{\substack{P\in P_\alpha\\ Q\in P_\beta}}\{\chi(\bar{z}'_{P^{-1}(1)}z_{Q^{-1}(1)})\chi(\bar{z}'_{P^{-1}(1)}\bar{z}'_{P^{-1}(2)}z_{Q^{-1}(1)}z_{Q^{-1}(2)})\cdots$$
$$\times\chi(\bar{z}'_{P^{-1}(1)}\cdots\bar{z}'_{P^{-1}(N-1)}z_{Q^{-1}(1)}\cdots z_{Q^{-1}(N-1)})$$
$$\times\delta(\bar{z}'_1\cdots\bar{z}'_N\ z_1\cdots z_N)\overline{\exp[-i\varphi_P(\mathbf{z}'_\alpha)]}, \tag{3.5}$$

where z'_i is parametrized by the binding variables of α, the bar denotes complex conjugation, and

$$\chi(z)\equiv\lim_{\gamma\downarrow0}(z-1+\gamma)^{-1},\quad \delta(z)\equiv\frac{1}{2\pi}\lim_{\gamma\downarrow0}\frac{-2\gamma}{(1-z)^2-\gamma^2} \tag{3.6}$$

in the sense of distributions. Thus Eq. (3.4) can be written symbolically as

$$A_t^0(f,g)=2\pi\sum_{\substack{P\in P_\alpha\\ Q\in P_\beta}}\int\bar{f}\,(\Pi\chi_{PQ})\delta g\exp(-i\varphi_P)$$
$$\times\exp[it(\epsilon_\alpha-\epsilon_\beta)]\mu_\alpha\mu_\beta d\mathbf{z}'_\alpha d\mathbf{z}_\beta \tag{3.7}$$

(cf. Ref. 4, Eq. 3.1.4). In the proof of Lemma 3.1.4 of Ref. 4 the individual terms on the rhs of Eq. (3.7) were shown to vanish in the limit $t\to\pm\infty$ by a Riemann Lebesgue argument. Thus $\lim_{t\to\pm\infty}A_t^0(f,g)=0$ and the first part of Theorem 1 follows.

Proceeding to the computation of $\tilde{W}_\pm(\beta)$, we let g be as above. Then

$$W_\pm(\beta)g(\mathbf{z}_\beta)$$
$$=\lim_{t\to\pm\infty}\sum_{\mathbf{m}\in\mathbf{Z}^N}\int_{\Gamma_\beta}\bar{\psi}_\beta(\mathbf{z}_\beta,\mathbf{m})J\psi_\beta^0(\mathbf{z}'_\beta,\mathbf{m})$$
$$\times\exp\{it[\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)]\}g(\mathbf{z}'_\beta)\mu_\beta(\mathbf{z}'_\beta)d\mathbf{z}'_\beta$$
$$=2\pi\sum_{P,Q\in_\beta}\overline{\exp(-i\varphi_P)(\mathbf{z}_\beta)}\lim_{t\to\pm\infty}\int_{\hat{\Gamma}_\beta}(\chi(\bar{z}_{P^{-1}(1)}z'_{Q^{-1}(1)})$$
$$\times\cdots\chi(\bar{z}_{P^{-1}(1)}\cdots\bar{z}_{P^{-1}(N-1)}\ z'_{Q^{-1}(1)}\cdots z'_{Q^{-1}(N-1)})$$
$$\times\delta(\bar{z}_1\cdots\bar{z}_N\ z'_1\cdots z'_N)\exp\{it[\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(z'_\beta)]\}$$
$$\times g(\mathbf{z}'_\beta)\mu_\beta(\mathbf{z}'_\beta)d\mathbf{z}'_\beta. \tag{3.8}$$

We consider the individual terms, labeled by P, Q, on the rhs of Eq. (3.8). We say that Q fills its complexes successively if $Q^{-1}(i+1)=Q^{-1}(i)+1$ for each i if $Q^{-1}(i)\neq N_{jk}+j$ for some jk. Now if $P\neq Q$ or $P=Q$ but Q does not fill its complexes successively, then, for almost every $\mathbf{z}_\beta$, at most $r-1$ of the χ or δ factors are simultaneously singular (χ or δ is singular when its argument is one) as $\mathbf{z}'_\beta$ ranges over the support of g. Here, r is the number of complexes of the binding β. For such terms, one makes a change of variable, treating $\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)$ as an independent variable. One then performs the integration with respect to the remaining variables to get an L^1 function of $\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)$, which, on integration against $\exp\{it[\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)]\}$, vanishes for $t\to\pm\infty$ by the Riemann–Lebesgue Lemma. Further details of the argument are contained in the text following Lemma (3.1.4) of Ref. 4.

Thus Eq. (3.8) is equal to

$$2\pi\sum_{Q\in P'_\beta}\overline{\exp(-i\varphi_Q)(\mathbf{z}_\beta)}\lim_{t\to\pm\infty}\int_{\hat{\Gamma}_\beta}(\chi(\bar{z}_{Q^{-1}(1)}z'_{Q^{-1}(1)})$$
$$\times\cdots\chi(\bar{z}_{Q^{-1}(1)}\cdots\bar{z}_{Q^{-1}(N-1)}z'_{Q^{-1}(1)}\cdots z'_{Q^{-1}(N-1)})$$
$$\times\delta(\bar{z}_1\cdots\bar{z}_N\ z'_1\cdots z'_N)\exp\{it[\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)]\}$$
$$\times g(\mathbf{z}'_\beta)\mu_\beta(z'_\beta)dz'_\beta, \tag{3.9}$$

where P'_β denotes the successive permutations of P_β. The limit in this expression is evaluated in the appendix (Lemma A1) so that (3.9) is equal to

$$g(\mathbf{z}_\beta)\sum_{Q\in P'_\beta}\overline{\exp(-i\varphi_Q)}\prod_{jk<j'k'}\vartheta(\pm(\nu_{Q^{-1}(j'k')}-\nu_{Q^{-1}(jk)}))$$
$$=\left(\prod_{\substack{jk<j'k'\\ \nu_{jk}\gtrless\nu_{j'k'}}}\exp(i\varphi_{jk,j'k'})\right)g(\mathbf{z}_\beta)\ \text{a.e.}, \tag{3.10}$$

which concludes the proof of Theorem 1. ∎

APPENDIX

The objective of this appendix is to compute the limit encountered in Sec. 3. Let Q be a successive permutation (see Sec. 3); then Q has the effect of permuting the subintervals I_{jk}, and so we write, e.g. $j'k'=Q^{-1}(jk)$. By $jk<j'k'$ we mean $j<j'$ or $j=j'$ but $k<k'$. Let s be the union of all codimension one hypersurfaces in $\hat{\Gamma}_\beta$ of the form $\{\mathbf{z}_\beta | z_{jk}=1$ for some $jk\}$ and $\{\mathbf{z}_\beta | s_{jk}(\mathbf{z}_\beta)=0\}$, where $s_{jk}(\mathbf{z}_\beta)$ is the $r\times r$ Jacobian (r is the number of complexes),

$$\det\begin{vmatrix} \nabla_{\mathbf{z}_\beta} & \epsilon_\beta(\mathbf{z}_\beta) \\ \nabla_{\mathbf{z}_\beta} & z_{11} \\ & \cdot \\ & \cdot \\ & \cdot \\ (\nabla \mathbf{z}_\beta & z_{jk})^{\wedge} \\ & \cdot \\ & \cdot \\ & \cdot \end{vmatrix}, \tag{A1}$$

with $(\nabla_{\mathbf{z}_\beta}\ z_{jk})^{\wedge}$ denoting deletion of the row $\nabla_{\mathbf{z}_\beta}\ z_{jk}$; $\nabla_{\mathbf{z}_\beta}$ is the gradient with respect to $\mathbf{z}_\beta$.

Lemma A.1: Let Q be a successive permutation and let $g(\mathbf{z}_\beta)$ be a C^∞ function with support bounded away from the set s. Then

$$\begin{aligned} L_\pm(\mathbf{z}_\beta, Q) &\equiv 2\pi \lim_{t\to\pm\infty} \int_{\hat{\Gamma}_\beta} \{\chi(\bar{z}_{Q^{-1}(1)} z'_{Q^{-1}(1)}) \\ &\times \chi(\bar{z}_{Q^{-1}(1)}\bar{z}_{Q^{-1}(2)} z'_{Q^{-1}(1)} z'_{Q^{-1}(2)}) \cdots \\ &\times \chi(\bar{z}_{Q^{-1}(1)}\cdots\bar{z}_{Q^{-1}(N-1)}\ z'_{Q^{-1}(N-1)}\cdots z'_{Q^{-1}(N-1)}) \\ &\times \delta(\bar{z}_1\cdots\bar{z}_N\ z'_1\cdots z'_N)\exp\{it[\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)]\} \\ &\times g(\mathbf{z}'_\beta)\mu_\beta(\mathbf{z}'_\beta)d\mathbf{z}'_\beta\} \\ &= \prod_{jk<j'k'} \vartheta(\pm(\nu_{Q^{-1}(j'k')}-\nu_{Q^{-1}(jk)}))\ \text{a.e.}, \end{aligned} \tag{A2}$$

where

$$\nu_{jk} \equiv i\left(\frac{jz_{jk}-j+1}{j}\ \frac{\partial\epsilon_\beta(\mathbf{z}_\beta)}{\partial z_{jk}}\right) \tag{A3}$$

and in Eq. (A2) it is understood that

$$z_{N_{jk}+j-l} = l^1(z_{jk}),\quad z'_{N_{jk}+j-l} = l^1(z'_{jk})\ \text{for}\ 0\le l<j.$$

The distributions χ, δ are defined in Eq. (3.6).

Proof: We consider only the case where Q is the identity permutation; the case of an arbitrary permutation involves only an elementary permutation of indices.

The first observation is that, although there are N χ and δ factors in the integrand, only r of these are singular in the sense that their arguments can equal one. The remaining factors have arguments of modulus strictly greater than one, by the restriction on the support of g. The (potentially) singular factors are the ones with arguments

$$\bar{z}_1\cdots\bar{z}_{N_{jk}+j}\ \ z'_1\cdots z'_{N_{jk}+j}\ \text{for some}\ jk.$$

The second observation is that the limit is zero if g vanishes at z_β for almost all z_β. The proof of this assertion is a Riemann–Lebesgue argument; one makes a change of variable with $\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)$ regarded as one of the independent variables. (The change of variable is permitted by the restriction on supp g.) One then does the integral with respect to the remaining variables to get an L^1 function, which on integration against $\exp\{it[\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)]\}$ vanishes for $t\to\pm\infty$ (cf. Ref. 4, Lemma 3.1.4). If $g(\mathbf{z}_\beta)\ne 0$, this observation allows one to replace $g(\mathbf{z}'_\beta)$ by $g(\mathbf{z}_\beta)\varphi(\mathbf{z}'_\beta)$, with $\varphi(\mathbf{z}'_\beta)C^\infty$ equal to 1 at $\mathbf{z}_\beta$ and having support $\subset$ supp g, and still obtain the same limit. (A particular φ will be constructed later.) The observation can also be applied to the nonsingular χ factors and μ_β. Thus $L_\pm(\mathbf{z}_\beta, 1)$ can be written

$$\begin{aligned} L_\pm(\mathbf{z}_\beta, 1) &= 2\pi g(\mathbf{z}_\beta) \\ &\times \prod_{\substack{i\ne N_{jk}+j \\ \text{for any } jk}} \chi(\bar{z}_1\cdots\bar{z}_i\ z_1\cdots z_i) \\ &\times\left\{\prod_j\prod_k^{n_j}\left[\frac{(-1)^{j-1}}{(2\pi i)}((j-1)!)^2\prod_{l=1}^{j-1}\left(\frac{z_{jk}-1}{lz_{jk}-l+1}\right)^2\right]\right\} \\ &\times \lim_{t\to\pm\infty} \prod_{\substack{i=N_{jk}+j \\ i=N}} \chi(\bar{z}_1\cdots\bar{z}_i\ z'_1\cdots z'_i) \\ &\times \delta(\bar{z}_1\cdots\bar{z}_N\ z'_1\cdots z'_N)\exp\{it[\epsilon_\beta(\mathbf{z}_\beta)-\epsilon_\beta(\mathbf{z}'_\beta)]\} \\ &\times \varphi(\mathbf{z}_\beta)\prod_j\prod_k^{n_j}\frac{j\,dz'_{jk}}{(jz'_{jk}-j+1)}, \end{aligned} \tag{A4}$$

where we have used Eq. (2.5).

The factors before the limit on the rhs of Eq. (A4) can be simplified. On $\hat{\Gamma}_\beta$, $z_{N_{jk}+1}\cdots z_{N_{jk}+j}$ is of unit modulus,

$$z_{N_{jk}+1}\cdots z_{N_{jk}+j-l} = (jz_{jk}-j+1)/[(j-l)z_{jk}-j+l+1],$$

and

$$\bar{z}_{N_{jk}+1}\cdots\bar{z}_{N_{jk}+j-l} = (lz_{jk}-l+1)^{-1}$$

so that

$$\prod_{i=N_{jk}+1}^{N_{jk}+j-1}\chi(\bar{z}_1\cdots\bar{z}_i\ z_1\cdots z_i) = \frac{(-1)^{j-1}}{[(j-1)!]^2}\prod_{l=1}^{j-1}\left(\frac{lz_{jk}-l+1}{z_{jk}-1}\right)^2. \tag{A5}$$

Thus Eq. (A4) is equal to

$$g(\mathbf{z}_\beta)\lim_{t\to\pm\infty}\frac{2\pi}{(2\pi i)^r}$$

$$\times\int_{|x_i|=1}\prod_{i<r}\chi\left(\frac{x_1'\cdots x_i'}{x_1\cdots x_i}\right)\delta\left(\frac{x_1'\cdots x_r'}{x_1\cdots x_r}\right)$$

$$\times\exp\{it[\epsilon_\beta(\mathbf{x})-\epsilon_\beta(\mathbf{x}')]\}\varphi(\mathbf{x}')\frac{d\mathbf{x}'}{x_1'\cdots x_r'}, \qquad \text{(A6)}$$

where, within the integral, we have made the substitution of variables $\mathbf{x}=(x_1,\ldots,x_r)$ with $x_i=(jz_{jk}-j+1)$ for $i=k+\sum_{j'=1}^{j-1}n_{j'}$. Here $\epsilon_\beta(\mathbf{x})=\sum_{i=1}^{r}\epsilon_i(x_i)$ with

$$\epsilon_i(x_i)=-\frac{(x_i-1)^2}{2jx_i},\quad i=k+\sum_{j'=1}^{j-1}n_j', \qquad \text{(A7)}$$

the energy of the jth complex parametrized by x_i.

We now make the substitution of variable $y_i=x_1x_2\cdots x_i$, $i=1,\ldots,r$ to obtain

$$L_\pm(z_\beta,1)$$

$$=g(\mathbf{z}_\beta)\lim_{t\to\pm\infty}[2\pi/(2\pi i)^r]$$

$$\times\int_{|y_i|=1}(y_1\cdots y_{r-1}/y_1'\cdots y_{r-1}')\prod_{i<r}[y_i'-(1-0)y_i]^{-1}$$

$$\times\delta(y_r'/y_r)\exp\{it[\epsilon_\beta(\mathbf{y})-\epsilon_\beta(\mathbf{y}')]\}\varphi(\mathbf{y}')d\mathbf{y}'/y_r'$$

$$=g(\mathbf{z}_\beta)\lim_{t\to\pm\infty}\frac{1}{(2\pi i)^{r-1}}$$

$$\times\int_{|y_i|=1}\left(\prod_{i<r}(y_i'-(1-0)y_i)^{-1}\right.$$

$$\times\exp\{it[\epsilon_\beta(\mathbf{y})-\epsilon_1(y_1')-\epsilon_2(y_2')\cdots-\epsilon_r(y_r/y_{r-1}')]\}$$

$$\left.\times\varphi(y_1',\ldots,y_{r-1}',y_r)\,dy_1'\cdots dy_{r-1}'\right), \qquad \text{(A8)}$$

where in the last step we have used the Riemann–Lebesgue argument which effectively evaluates $(y_1\cdots y_{r-1})/(y_1'\cdots y_{r-1})$ at $y_1'\cdots y_{r-1}'=y_1\cdots y_{r-1}$ and we have done the y_r' integration, using the fact that

$$\int_{|z|=1}\delta(z)h(z)dz=ih(1). \qquad \text{(A9)}$$

In Eq. (A8), $\epsilon_\beta(\mathbf{y})\equiv\sum_{i=1}^{r}\epsilon_i(y_i/y_{i-1})$, with $y_0\equiv 1$.

We next consider just the y_{r-1}' integration, first making yet another change of variable, while imposing some conditions on φ. Let $\mu=\epsilon_r(y_r/y_{r-1})-\epsilon_r(y_r/y_{r-1}')+\epsilon_{r-1}(y_{r-1}/y_{r-2}')-\epsilon_{r-1}(y_{r-1}'/y_{r-2}')$. (The transformation $y_{r-1}'\to\mu$ is locally 1–1 about $\mathbf{y}$ for almost all $\mathbf{y}$ and hence $\mathbf{z}_\beta$.) Assume φ is of the form $\varphi(y_1'\cdots y_{r-1}',y_r)=t(y_1',\ldots,y_{r-2}')s(y_{r-2}',y_{r-1}')$ with t as yet arbitrary but s given implicitly below (y_r is treated as constant). We have that

$$(y_{r-1}'-(1-\gamma)y_{r-1})^{-1}$$

$$=\left(\gamma y_{r-1}+\mu\,\frac{\partial y_{r-1}'}{\partial\mu}(y_{r-2}',y_{r-1})\right)^{-1}[1+v(\gamma,y_{r-2}',\mu)] \qquad \text{(A10)}$$

with $v=0(\mu)$ uniformly in γ, so that in terms of this new variable

$$L_\pm(\mathbf{z}_\beta,1)$$

$$=\frac{g(\mathbf{z}_\beta)}{(2\pi i)^{r-1}}\lim_{t\to\pm\infty}\int\prod_{i=1}^{r-2}[y_i'-(1-0)y_i]^{-1}M(t)$$

$$\times\exp\{it[\epsilon_1(y_1)+\cdots\epsilon_{r-1}(y_{r-1}/y_{r-2})-\epsilon_1(y_1)-\cdots$$

$$-\epsilon_{r-2}(y_{r-2}'/y_{r-3}')-\epsilon_{r-1}(y_{r-1}/y_{r-2}')]\}t(y_1',\ldots,y_{r-2}')$$

$$\times dy_1'\cdots dy_{r-2}', \qquad \text{(A11)}$$

where

$$M(t)$$

$$=\lim_{\gamma\downarrow 0}\int\frac{d\mu\, e^{it\mu}[1+v(\gamma,y_{r-2}',\mu)]s(y_{r-2}',y_{r-1}')}{(\partial\mu/\partial y_{r-1}')(y_{r-2}',y_{r-1}')[\gamma y_{r-1}+\mu(\partial y_{r-1}'/\partial\mu)(y_{r-2}',y_{r-1})]}$$

$$=\lim_{\gamma\downarrow 0}\int\frac{d\mu\, e^{it\mu}}{[\mu+\gamma y_{r-1}(\partial\mu/\partial y_{r-1}')(y_{r-2}',y_{r-1})]}$$

$$\times\left[\frac{(\partial\mu/\partial y_{r-1}')(y_{r-2}',y_{r-1})[1+v(0,y_{r-2}',\mu)s(y_{r-2}',y_{r-1}')]}{(\partial\mu/\partial y_{r-1}')(y_{r-2}',y_{r-1}')}\right]. \qquad \text{(A12)}$$

Now we simply define s in such a manner that the quantity within the large brackets in the integrand of (A1) is in fact only a C^∞ function of μ which is identically one in a neighborhood of $\mu=0$. [Note $s(y_{r-2},y_{r-1})=1$]. Denote this function $\omega(\mu)$. Then it is straightforward to show by complex integration that

$$M(t)=\lim_{\gamma\downarrow 0}$$

$$\times\int\frac{e^{it\mu}\omega(\mu)d\mu}{(\mu+\gamma y_{r-1}(\partial\mu/y_{r-1}')(y_{r-2}',y_{r-1})}$$

$$=2\pi i\vartheta(ity_{r-1}(\partial\mu/\partial y_{r-1}')(y_{r-2}',y_{r-1}))+O(1/t)$$

$$=2\pi i\vartheta(t(\nu_r-\nu_{r-1}))+O(1/t), \qquad \text{(A13)}$$

for y_{r-2}' near y_{r-2} and the term $O(1/t)$ independent of y_{r-2}'. Here,

$$\nu_l=i\frac{y_l}{y_{l-1}}\frac{\partial\epsilon_l}{\partial x_l}\left(\frac{y_l}{y_{l-1}}\right),\quad l=1,2,\ldots,r. \qquad \text{(A14)}$$

Now note that the integral (A11) is precisely of the form (A8) with r diminished by one [and $M(t)$ independent of y_{r-2}' for y_{r-2}' near y_{r-2}]. Hence the argument may be repeated for y_{r-2}', y_{r-3}', etc., and we obtain

$$L_\pm(\mathbf{z}_\beta, 1) = g(\mathbf{z}_\beta) \prod_{l=2}^{r} \vartheta(\pm(\nu_l - \nu_{l-1}))$$

$$\equiv g(\mathbf{z}_\beta) \prod_{i<j} \vartheta(\pm(\nu_j - \nu_i)). \tag{A15}$$

Taking an arbitrary successive permutation, and writing ν_l in terms of $\mathbf{z}_\beta$, we obtain the lemma. ∎

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Conditional probabilities and statistical independence in quantum theory

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The problem of defining conditional probabilities and the notion of statistical independence in quantum theory is analyzed. It is shown that (unlike in classical probability theory) the conditional probabilities of a given set of events can be determined only if the sequence of all the experiments performed on the system is also specified. Such a specification is necessary also for the concept of statistical independence to become physically meaningful.

Recent investigations[1-3] have shown that the statistics of successive observations in quantum theory should be studied in a framework of quantum probability theory. In the present investigation we shall examine the problem of defining conditional probabilities, and also the related notion of statistical independence for experiments performed on a quantum system. First, we define operationally meaningful joint—and conditional probabilities of a set of events when a given sequence of experiments in performed on the system. We shall demonstrate that *these probabilities depend not only on the set of events considered, but also on the sequence of experiments performed on the system*. This, in fact, is the essential physical content of the various nonclassical properties of the quantum theoretic probabilities, which are sometimes collectively referred to as the "quantum interference of probabilities."[2,4]

We shall also show that an operationally meaningful notion of statistical independence can be formulated for a set of experiments, or the corresponding random variables; however, such a notion cannot be defined just for a given set of events alone, as is possible in classical probability theory. Once the notion of statistical independence is clearly formulated, one can proceed to a study of situations where there exists some kind statistical dependence between the set of experiments performed on the system. In order to illustrate this point, we finally make a few remarks on quantum Markov chains defined on a discrete value space.

1. PRELIMINARIES

Let $(\Omega, \mathcal{J}, \mu)$ be a classical probability space.[5] Let $\{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m\}$ be a set of events. Then the joint probability $\Pr\{A_1, A_2, \ldots, A_n\}$ for the set of events $\{A_i\}$ $(1 \leq i \leq n)$ to be observed, and the conditional probability $\Pr\{A_1, A_2, \ldots, A_n/B_1, B_2, \ldots, B_m\}$ for the set of events $\{A_i\}$ $(1 \leq i \leq n)$ to be observed, given that the set of events $\{B_\gamma\}$ $(1 \leq \gamma \leq m)$ are observed, are defined by the following equations:

$$\Pr\{A_1, A_2, \ldots, A_n\} = \mu(A_1 \cap A_2 \cap \cdots \cap A_n), \tag{1.1}$$

$$\Pr\{A_1, A_2, \ldots, A_n/B_1, B_2, \ldots, B_m\} = \mu(B_1 \cap B_2 \cap \cdots \cap B_m \cap A_1 \cap \cdots \cap A_n)/\mu(B_1 \cap \cdots \cap B_m). \tag{1.2}$$

It is clear that the joint probabilities (1.1) and the conditional probabilities (1.2), depend only on the set of events $\{A_1, A_2, \ldots, A_n, B_1, \ldots, B_m\}$ considered. One need not make any reference at all to the set of experiments in which these events have been observed—or equivalently, to the set of random variables that are being considered.[6]

In order to study the joint and conditional probabilities in quantum theory, we shall make use of the framework of quantum probability theory outlined in Ref. 2. In quantum theory there is, corresponding to each event that a particular outcome is observed in a given experiment, an associated "measurement transformation" or operation. The set of all operations constitutes the event space, which has now a structure quite different from that of a Boolean σ-algebra. Here, we shall outline only some of the salient features of quantum probability theory (mainly to set up the notation), and refer the reader to Ref. 2 for a detailed exposition.

Let V be the ordered Banach space (under the trace norm) of all self-adjoint trace-class operators on a Hilbert space $\mathcal{H}$. Then the set of all operations $\mathcal{O}$, is the set of all positive, norm-nonincreasing linear transformations on V. $\mathcal{O}$ is partially ordered by the relation

$$\mathcal{E}_1 \leq \mathcal{E}_2 \Longleftrightarrow \mathcal{E}_1(v) \leq \mathcal{E}_2(v), \tag{1.3}$$

for all $v \in V^+$. $\mathcal{O}$ has a subset Σ (consisting, in general, of more than one element) of maximal elements, which have the property

$$\xi \in \Sigma \Longleftrightarrow \mathrm{Tr}[\xi(v)] = \mathrm{Tr}[v] \tag{1.4}$$

for all $v \in V^+$. In $\mathcal{O}$, the conjunction of two events $\mathcal{E}_1$, $\mathcal{E}_2$ is given by

$$(\mathcal{E}_1 \wedge \mathcal{E}_2)(v) = \mathcal{E}_2(\mathcal{E}_1(v)), \tag{1.5}$$

for all $v \in V$; this corresponds to the event that both the events $\mathcal{E}_1$ and $\mathcal{E}_2$ are observed in that order. If $\mathcal{E}_1 + \mathcal{E}_2 \in \mathcal{O}$ also, then we say that the events $\mathcal{E}_1$ and $\mathcal{E}_2$ are mutually disjoint and define their disjunction $\mathcal{E}_1 \vee \mathcal{E}_2$ by the relation

$$\mathcal{E}_1 \vee \mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_2. \tag{1.6}$$

Given an event $\mathcal{E} \in \mathcal{O}$, there are in general *several complementary events* $\bar{\mathcal{E}} \in \mathcal{O}$, such that $\mathcal{E} \vee \bar{\mathcal{E}} \in \Sigma$. Hence, given any $\mathcal{E} \in \mathcal{O}$, *there exist several maximal events* $\xi \in \Sigma$ such that $\mathcal{E} \leq \xi$.

By a quantum probability space we mean an ordered

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pair $(\mathcal{O}, \mu)$ where the state μ is a strongly continuous linear mapping from $\mathcal{O}$ into $[0,1]$, which satisfies[7]

$$\mu(\xi)=1, \tag{1.7}$$

for all $\xi \in \Sigma$ and

$$\mu(\mathcal{E} \wedge \xi)=\mu(\mathcal{E}) \tag{1.8}$$

for all $\xi \in \Sigma$ and $\mathcal{E} \in \mathcal{O}$. In particular, there are states $\{\mu_\rho\}$ which can be specified by the density operators $\{\rho\}$, (i.e., $\rho \in V^+$ and $\mathrm{Tr}\rho=1$), in the following way:

$$\mu_\rho(\mathcal{E})=\mathrm{Tr}[\mathcal{E}(\rho)], \tag{1.9}$$

for all $\mathcal{E} \in \mathcal{O}$.

An experiment X whose outcomes all lie in the value space R (in general, a complete separable metric space) is specified by the corresponding random variable (which shall also be denoted as X), which is a σ-additive map from $\mathcal{B}(R)$ into $\mathcal{O}$, such that $X(R) \in \Sigma$. $X(E)$ corresponds to the event that the outcome of the experiment X is found to lie in the Borel set $E \in \mathcal{B}(R)$; $\mu(X(E))$ will be the probability for this event to be observed when the experiment X is performed on a system in state μ.

In this paper we shall employ the Heinsenberg picture of evolution—i.e., under a time evolution only the random variables are assumed to evolve with time. Hence, if we say that an experiment X_1 was conducted at time t_1 and an experiment X_2 at time t_2 (with $t_1 < t_2$), it is to be understood that the time evolution up to the time t_i is already taken into account in the specification of the random variable X_i. Then $\mu(X_1(E_1) \wedge X_2(E_2))$ will be the probability for the event that the experiment X_1 (at t_1) yields a value in E_1 and the *next* experiment X_2 (at t_2) yields a value in E_2 when a system in state μ is subjected to the sequence of experiments $\{X_1, X_2\}$.

We have so far summarized only the formal structure of quantum probability theory. There still remains the question as to what meaning is to be attached to statements such as "the probability for observing the event... is...." This question is definitely of great importance as it could possibly be said[8–10] That the differences between the various interpretations of quantum theory essentially arise out of the differences in the interpretations of the probabilities[11] predicted by the theory. For the purposes of the present paper we shall adopt the following "relative frequency" interpretation of the probabilities such as $\mu(X_1(E_1) \wedge X_2(E_2) \wedge \cdots \wedge X_r(E_r))$:

Let

$$n^{\mu,N}_{X_1,X_2,\ldots,X_r}\{X_1(E_1), X_2(E_2), \ldots, X_r(E_r)\}$$

be the number of systems for which the outcome of the experiment X_i is found to be in E_i for all $1 \le i \le r$, when N systems all in state μ, are separately subjected to the sequence of experiments $X_1, X_2, \ldots X_r$. Then we shall make the following identification:

$$\mu(X_1(E_1) \wedge X_2(E_2) \wedge \cdots \wedge X_r(E_r)) = \lim_{N\to\infty} n^{\mu,N}_{X_1,X_2,\ldots,X_r}\{X_1(E_1), X_2(E_2), \ldots, X_r(E_r)\}/N. \tag{1.10}$$

It should, of course, be noted that the identification of probabilities with relative frequencies gives rise to several difficulties (both in the definition and measurement of probabilities), which have not been completely resolved even in the context of classical probability theory.[12] However, for the purposes of the present paper, the identification (1.10) is sufficient to provide operationally meaningful definitions of conditional probabilities and the notion of statistical independence.

2. CONDITIONAL PROBABILITIES

We shall consider the following general situation where a sequence of experiments $\{X_1, X_2, \ldots, X_r\}$ are performed on a system in state μ at times $t_1 < t_2 < \cdots < t_r$. For simplicity, we shall assume that all these experiments have the same value space R and write

$$X_i(R)=\xi_i \in \Sigma. \tag{2.1}$$

If $\{i_1, i_2, \ldots, i_k\}$ is a subset of $\{1,2,\ldots,r\}$, with elements all distinct,[13] and $\{E_{i_1}, E_{i_2}, \ldots, E_{i_k}\} \subset \mathcal{B}(R)$, we shall denote by $\mathrm{Pr}_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots X_{i_k}(E_{i_k})\}$ the joint probability that the outcome of the experiment X_{i_α} is found to lie in $E_{i_\alpha}(1 \le \alpha \le k)$, when the system is subjected to the sequence of experiments $\{X_1, X_2, \ldots, X_r\}$. This will be operationally identified with

$$\lim_{N\to\infty} n^{\mu,N}_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\}/N, \tag{2.2}$$

where $n^{\mu,N}_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\}$ is the number of systems for which the outcome of X_{i_α} is found to lie in E_{i_α} $(\alpha=1,2,\ldots,k)$, when a total of N systems, all in state μ, are subjected to the sequence of experiments $\{X_1, X_2, \ldots, X_r\}$. It is clear that $n^{\mu,N}_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\}$ is the same as the number of systems for which the outcome of X_{i_α} is found to lie in E_{i_α} $(\alpha=1,2,\ldots,k)$, and the outcome of the rest of the experiments $\{X_\beta\}$ lies anywhere in R, for each $\beta \in \{1,2,\ldots,r\}\backslash\{i_1,\ldots,i_k\}$.

If π is the permutation of the indices $\{i_1, i_2, \ldots, i_k\}$ such that $\pi i_1 < \pi i_2 < \cdots < \pi i_k$, and if we write

$$\pi i_\alpha = p_\alpha \quad (1 \le \alpha \le k), \tag{2.3}$$

then, from our remarks above and the basic prescriptions of quantum probability theory (outlined at the end of Sec. 1), we can conclude that the limit (2.2) is nothing but

$$\mu(X_1(R) \wedge X_2(R) \wedge \cdots \wedge X_{p_1-1}(R) \wedge X_{p_1}(E_{p_1}) \wedge X_{p_1+1}(R) \wedge \cdots \wedge X_{p_k-1}(R) \wedge X_{p_k}(E_{p_k}) \wedge X_{p_k+1}(R) \wedge \cdots \wedge X_r(R)). \tag{2.4}$$

By using (2.1) and (1.8), we obtain from (2.2), (2.4) the equation

$$\mathrm{Pr}_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\} = \mu(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{p_1-1} \wedge X_{p_1}(E_{p_1}) \wedge \xi_{p_1+1} \wedge \cdots \wedge \xi_{p_k-1} \wedge X_{p_k}(E_{p_k})), \tag{2.5}$$

where, as we noted earlier, $p_\alpha = \pi i_\alpha$ $(1 \le \alpha \le k)$ and π is the permutation of the set of indices $\{i_1, \ldots, i_k\}$ such that $\pi i_1 < \pi i_2 < \cdots < \pi i_k$.

We now proceed to the operational definition of conditional probabilities. Let $\{i_1, i_2, \ldots, i_k, \gamma_1, \ldots, \gamma_l\}$ be a subset of $\{1,2,\ldots, r\}$ with elements all distinct.[13] We shall denote by

$$\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k})/X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l})\}$$

the conditional probability that the outcome of X_{i_α} is found to lie in E_{i_α} $(1 \le \alpha \le k)$, given that the outcome of X_{γ_β} is found to lie in E_{γ_β} $(1 \le \beta \le l)$, when the system in state μ is subjected to a sequence of experiments $\{X_1, X_2, \ldots, X_r\}$. This shall be operationally identified with

$$\lim_{N\to\infty}\Big([n^{\mu,N}_{X_1,X_2,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), X_{\gamma_2}(E_{\gamma_2}), \ldots, X_{\gamma_l}(E_{\gamma_l}), X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k})\}/N] \times [n^{\mu,N}_{X_1,X_2,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l})\}/N]^{-1}\Big). \tag{2.6}$$

Hence we conclude that

$$\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k})/X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l})\} = \frac{\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k}), X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l})\}}{\Pr_{X_1,X_2,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l})\}} \tag{2.7}$$

where the numerator and the denominator on the right-hand side can be obtained by a relation like (2.5). From (2.5) we can also conclude that

$$\begin{aligned}
&\Pr_{X_1,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), X_{\gamma_2}(E_{\gamma_2}), \ldots, X_{\gamma_l}(E_{\gamma_l})\}\\
&= \Pr_{X_1,X_2,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), X_{\gamma_2}(E_{\gamma_2}), \ldots, X_{\gamma_l}(E_{\gamma_l}), X_{i_1}(R), \ldots, X_{i_k}(R)\}\\
&= \Pr_{X_1,X_2,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l}), X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k})\}\\
&\quad + \Pr_{X_1,X_2,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l}), X_{i_1}(\bar{E}_{i_1}), \ldots, X_{i_k}(\bar{E}_{i_k})\}\\
&\ge \Pr_{X_1,X_2,\ldots,X_r}\{X_{\gamma_1}(E_{\gamma_1}), X_{\gamma_2}(E_{\gamma_2}), \ldots, X_{\gamma_l}(E_{\gamma_l}), X_{i_1}(E_{i_1}),\\
&\quad \times X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\},
\end{aligned} \tag{2.8}$$

where $\bar{E}_{i_\alpha}$ is the unique complement of E_{i_α} in $\mathcal{B}(R)$ and we have used the σ-additivity of the random variables X_{i_α}. From (2.8), we note first that the numerator in (2.7) vanishes when the denominator does, and then we take the conditional probability (2.7) to be zero. We have therefore the relation

$$0 \le \Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})/X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l})\} \le 1. \tag{2.9}$$

In order to illustrate the essential nonclassical features of these conditional probabilities, we first consider the situation where just two experiments $\{X_1, X_2\}$ are performed at times t_1, t_2, with $t_1 < t_2$. We have then the following joint and conditional probabilities:

$$\Pr_{X_1,X_2}\{X_1(E_1), X_2(E_2)\} = \mu(X_1(E_1) \wedge X_2(E_2)); \tag{2.10a}$$

$$\Pr_{X_1,X_2}\{X_1(E_1)\} = \mu(X_1(E_1)); \tag{2.10b}$$

$$\Pr_{X_1,X_2}\{X_2(E_2)\} = \mu(\xi_1 \wedge X_2(E_2)); \tag{2.10c}$$

$$\Pr_{X_1,X_2}\{X_2(E_2)/X_1(E_1)\} = \frac{\mu(X_1(E_1)\wedge X_2(E_2))}{\mu(X_1(E_1))}; \tag{2.11a}$$

$$\Pr_{X_1,X_2}\{X_1(E_1)/X_2(E_2)\} = \frac{\mu(X_1(E_1)\wedge X_2(E_2))}{\mu(\xi_1 \wedge X_2(E_2))}. \tag{2.11b}$$

It is important to realize that

$$\Pr_{X_1,X_2}\{X_1(E_1)/X_2(E_2)\} \ne \frac{\mu(X_1(E_1)\wedge X_2(E_2))}{\mu(X_2(E_2))}. \tag{2.12a}$$

In fact the quantity on the right-hand side is not constrained to be less than unity and has no operational interpretation whatsoever as a probability. In the same way, we see from Eq. (2.10c) that in general

$$\Pr_{X_1,X_2}\{X_2(E_2)\} \ne \mu(X_2(E_2)). \tag{2.12b}$$

However, the quantity on the rhs of (2.12b) has an operational interpretation as the probability $\Pr_{X_2}\{X_2(E_2)\}$ for the outcome of X_2 to in E_2 when X_2 is the first experiment conducted on the system in state μ. Therefore, the inequality (2.12b) (which has sometimes been considered as an expression of a certain "quantum interference of probabilities"[2,4]) is equivalent to the following relation:

$$\Pr_{X_1,X_2}\{X_2(E_2)\} \ne \Pr_{X_2}\{X_2(E_2)\}. \tag{2.13}$$

This brings us to the central point that we want to emphasize which is that, *in quantum theory, for any given set of events, operationally meaningful joint and conditional probabilities can be (defined and) determined in general, only if the sequence of experiments performed on the system is also specified.* For example, the probabilities (2.10c) and (2.11b) depend not only on $X_1(E_1)$ and $X_2(E_2)$ (the events considered), but also on ξ_1. We have already noted in Sec. 1 that for any given event $\mathcal{E}_1$ [$X_1(E_1)$ say], the maximal element $\xi_1 \in \Sigma$ such that $\xi_1 \ge \mathcal{E}_1$ is not unique, but depends on the experiment (X_1) we have chosen to perform. This is because there are, in general several maximal elements $\xi_1 \in \Sigma$ which satisfy $\xi_1 \ge \mathcal{E}_1$, and the maximal element $X_1(R)$ can be any one of them.

In the general case we see from (2.5) and (2.7) that

$$\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\},$$

$$\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k})/X_{\gamma_1}(E_{\gamma_1}), \ldots, X_{\gamma_l}(E_{\gamma_l})\}$$

depend in general also on $\{\xi_1, \xi_2, \ldots, \xi_{r-1}\}$. From Eq. (1.8), we can easily show that *probabilities involving a given set of events depend only on those experiments which are performed prior to the time at which the last event (among the set of events considered) is observed.* For example, in Eq. (2.5), since p_k is the largest of $\{i_1, i_2, \ldots, i_k\}$, we have

$$\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k})\} = \Pr_{X_1,X_2,\ldots,X_{p_k}}\{X_{i_1}(E_{i_1}), \ldots, X_{i_k}(E_{i_k})\}. \tag{2.14}$$

In particular, the probabilities (2.10) and (2.11) would remain the same even in the case when arbitrary experiments $X_3, X_4, \ldots, X_r$, etc., are performed after X_1, X_2.

Equations (2.5), (2.7) are extremely general and can be used to determine operationally meaningful joint—and conditional probabilities for any set of events when an arbitrary sequence of measurements is performed on the quantum system. If, in particular, the times t_{i_α} $(1 \le \alpha \le k)$ in (2.7) are all greater (lesser) than t_{γ_β}

$(1 \leqslant \beta \leqslant l)$, then we have the case of prediction (retrodiction. Equation (2.7) can also be used in the general case when

(i) some of the $\{t_{\gamma_\beta}\}$ lie between different t_{i_α};

(ii) the set $\{i_1, i_2, \ldots, i_k, \gamma_1, \gamma_2, \ldots, \gamma_l\}$ is a proper subset of $\{1, 2, \ldots, r\}$—i.e., the events considered refer only to a proper subset of the set of all experiments performed.

We shall now briefly review some of the earlier investigations on conditional probabilities in quantum theory. At the outset, we may note that (barring Refs. 1,2) all such studies have been confined to a class of events which correspond to the so called "ideal measurements." An ideal measurement event may be represented as an operation $\mathcal{E}$ of the form

$$\mathcal{E}v = PvP, \tag{2.15}$$

for all $v \in V$, where P is a prejection operator on $\mathcal{H}$. As we have already mentioned, the nonclassical features of the quantum conditional probabilities for ideal measurement events were first noted by de Broglie.[4] Later, Watanabe[14] emphasized the fact that the purely retrodictive conditional probabilities (for which such nonclassical features were first discovered) were also physically very meaningful. For an elegent calculation of the purely retrodictive probability given by (2.11b) when only ideal measurements are considered, the reader is referred to the recent book of d'Espagnat.[15] Conditional probabilities for ideal measurement events which are not restricted to be either purely predictive or purely retrodictive were first considered by Aharanov, Bergman, and Lebowitz.[16] The general case referred to as (i) above, has been discussed in detail by Houtappel, Van Dam, and Wigner.[17]

Apart from the restrictions already mentioned, most of the above investigations also suffer from the following limitation: They also assume that the operation $\bar{\mathcal{E}}$ complementary to the operation $\mathcal{E}$ of Eq. (2.15) is always given by the relation

$$\bar{\mathcal{E}}v = (1 - P)\, v\, (1 - P), \tag{2.16}$$

for all $v \in V$, where 1 is the identity operator on $\mathcal{H}$. This is completely justified as long as the operation $\mathcal{E}$ corresponds to an event of the following form: "In an experiment to measure the observable represented (in the conventional framework of quantum theory) by the self-adjoint operator

$$A = \lambda_1 P + \lambda_2 (1 - P), \tag{2.17}$$

the outcome was found to be λ_1." However, we can also consider the operation $\mathcal{E}$ to be associated with the event "in a measurement of the observable represented by the self-adjoint operator

$$B = \mu_1 P + \mu_2 Q + \mu_3 R, \tag{2.18}$$

(where the projection operators Q, R are such that $Q + R = 1 - P$), the outcome was found to be μ_1." Then the complementary event $\bar{\mathcal{E}}$ is given by the relation

$$\bar{\mathcal{E}}v = QvQ + RvR, \tag{2.19}$$

which is completely different from (2,16). Therefore, it should be noted that, *even when we restrict ourselves to ideal measurement events [such as in Eq. (2.13)] only, the complementary event is not unique, and depends on the particular experiment (such as measurement of A or B above) performed on the system.*

We shall finally make a few remarks on the notion of conditional expectations in quantum theory. The notion of conditioning has been discussed in the lattice theoretical approach to quantum logic by Pool.[18] There have also been several investigations[19] on the definition of conditional expectations on Von Neumann algebras. However, as we have emphasized elsewhere,[2] both these approaches are not well-suited for a discussion of joint and conditional probabilities in quantum theory because, in these approaches, the mathematical characterization of an observable does not include a specification of the associated measurement transformations.

Recently Cycoon and Hellwig[20] have introduced a notion of "generalized conditional (GC) expectations" in the framework of quantum probability theory. They have also exhibited conditions under which the GC expectation corresponding to a given random variable X coincides with the dual mapping $X^*: \mathcal{B}(R) \to L(V', V')$, where V' is the dual of V, and, for each $E \in \mathcal{B}(R)$, $X^*(E)$ is the transposed operator of $X(E)$. However, the relation between these GC expectations (which have been introduced based on an anology with certain relations involving conditional expectations in classical probability theory) and the conditional probabilities is not transparent as in classical probability theory.

3. STATISTICAL INDEPENDENCE

The concept of statistical independence occupies a central position in probability theory. In fact, as Kolmogorov[21] has pointed out, "Historically, the independence of experiments and random variables represents the very mathematical concept that has given the theory of probability its peculiar stamp." In classical theory the notion of statistical independence can be defined for a given set of events as well as for a given set of experiments.[22] We shall now show that in quantum theory the notion of statistical independence can be defined in an operationally meaningful way only for sets of experiments (random variables), and that too only when the sequence of all the experiments performed on the system is also specified.

Let us again consider the general situation where a system in state μ is subjected to a sequence of experiments $\{X_1, X_2, \ldots, X_r\}$. It is operationally meaningful to say that the set of random variables $\{X_{i_1}, X_{i_2}, \ldots, X_{i_k}\}$ are statistical independent whenever

$$\lim_{N\to\infty} n^{\mu,N}_{X_1, X_2, \ldots, X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\}/N = \prod_{\alpha=1}^{k} \lim_{N\to\infty} n^{\mu,N}_{X_1, X_2, \ldots, X_r}\{X_{i_\alpha}(E_{i_\alpha})\}/N, \tag{3.1}$$

for all $E_{i_\alpha} \in \mathcal{B}(R)$. We are thus led to the following definition:

A set of random variables $\{X_{i_1}, X_{i_2}, \ldots, X_{i_k}\}$ is said to be statistically independent in the state μ when a sequence of experiments $\{X_1, X_2, \ldots, X_r\}$ are performed, iff

$$\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_1}(E_{i_1}), X_{i_2}(E_{i_2}), \ldots, X_{i_k}(E_{i_k})\}$$
$$=\prod_{\alpha=1}^{k}[\Pr_{X_1,X_2,\ldots,X_r}\{X_{i_\alpha}(E_{i_\alpha})\}], \tag{3.2}$$

for all $E_{i_\alpha} \in \mathcal{B}(R)$ $(1 \leq \alpha \leq k)$.

If in (3.2) we take some of the E_{i_α} to be R itself and use (2.5), we arrive at the result that any subset of a set of statistically independent random variables is also statistically independent, in the same state when the same sequence of experiments are considered.

From the above result and our definition (2.7) of conditional probabilities, we can establish the following conclusion (valid in classical theory also[19]):

A set of random variables $\{X_{i_1}, X_{i_2}, \ldots, X_{i_k}\}$ is satistically independent iff

$$\Pr_{X_1,X_2,\ldots,X_r}\{X_{q_l}(E_l)/X_{q_1}(E_1), X_{q_2}(E_2), \ldots, X_{q_{l-1}}(E_{l-1})\}$$
$$=\Pr_{X_1,X_2,\ldots,X_r}\{X_{q_l}(E_l)\}, \tag{3.3}$$

for all $E_\alpha \in \mathcal{B}(R)$ $(1 \leq \alpha \leq l)$, for all subsets $\{q_1, q_2, \ldots, q_l\}$ of the set of indices $\{i_1, i_2, \ldots, i_k\}$ for all $l \leq k$.

We shall now illustrate the above definition of statistical independence when just two experiments X_1, X_2 are performed at t_1, t_2 with $t_1 < t_2$. For simplicity, we assume that the value space is the set $\{0, 1\}$, and we write

$$X_i(\{1\}) = \mathcal{E}_i, \tag{3.4a}$$

$$X_i(\{0\}) = \bar{\mathcal{E}}_i, \tag{3.4b}$$

$$X_i(\{0,1\}) = \xi_i = \mathcal{E}_i + \bar{\mathcal{E}}_i, \tag{3.4c}$$

for $i = 1, 2$. Then X_1, X_2 are statistically independent in state μ iff the following conditions are satisfied:

$$\mu(\mathcal{E}_1 \wedge \mathcal{E}_2) = \mu(\mathcal{E}_1)\mu(\xi_1 \wedge \mathcal{E}_2), \tag{3.5a}$$

$$\mu(\mathcal{E}_1 \wedge \bar{\mathcal{E}}_2) = \mu(\mathcal{E}_1)\mu(\xi_1 \wedge \bar{\mathcal{E}}_2), \tag{3.5b}$$

$$\mu(\bar{\mathcal{E}}_1 \wedge \mathcal{E}_2) = \mu(\bar{\mathcal{E}}_1)\mu(\xi_1 \wedge \mathcal{E}_2), \tag{3.5c}$$

$$\mu(\bar{\mathcal{E}}_1 \wedge \bar{\mathcal{E}}_2) = \mu(\bar{\mathcal{E}}_1)\mu(\xi_1 \wedge \bar{\mathcal{E}}_2). \tag{3.5d}$$

However, it is sufficient to demand say

$$\mu(\mathcal{E}_1 \wedge \mathcal{E}_2) = \mu(\mathcal{E}_1)\,\mu(\xi_1 \wedge \mathcal{E}_2),$$

as (3.5b), (3.5c), and (3.5d) follow from (3.5a) once we use (3.4) and (1.8).

It should be remarked that unlike (3.5a), the equation

$$\mu(\mathcal{E}_1 \wedge \mathcal{E}_2) = \mu(\mathcal{E}_1)\,\mu(\mathcal{E}_2)$$

does not in general have any operational meaning of independence in quantum theory (unless, of course, $\xi_1 = I$, where I is the identity operation). From (3.5a) it is also clear that *even for just two events, the concept of statistical independence cannot be defined in quantum theory without making reference to the sequence of experiments performed.*

The operational definition of conditional probabilities and the concept of statistical independence that we have outlined is but the first step in the general analysis of statistical relations among a series of experiments performed on a quantum system. However, it is also an essential step before analyzing the various forms of statistical dependence that may exist between successive experiments. For example, we may consider a simple generalization of (3.3) which leads us to the following notion of a Markov chain.

A quantum Markov chain on a discrete value (state) space is a sequence $\{X_1, X_2, \ldots, X_r, \cdots\}$ of random variables defined on the value space $\{1, 2, \ldots, n\}$, which satisfy

$$\Pr_{X_1,X_2,\ldots,X_r,\ldots}\{X_r(\{\alpha_r\})/X_1(\{\alpha_1\}), X_2(\{\alpha_2\}), \ldots, X_{r-1}(\{\alpha_{r-1}\})\}$$
$$=\Pr_{X_1,X_2,\ldots,X_r,\ldots}\{X_r(\{\alpha_r\})/X_{r-1}(\{\alpha_{r-1}\})\}, \tag{3.6}$$

for all $\alpha_k \in \{1, 2, \ldots, n\}$ $(1 \leq k \leq r)$, for all $r \geq 2$

If we write

$$X_r(\{1, 2, \ldots, n\}) = \xi_r, \tag{3.7}$$

for all $r \geq 1$, then the singlet probabilities $p_r(\alpha_r)$ and the transition probabilities $p_{q,r}(\alpha_q, \alpha_r)$ are given by the relations

$$p_r(\alpha_r) = \Pr_{X_1, X_2, \ldots, X_r, \ldots}\{X_r(\{\alpha_r\})\}, \tag{3.8}$$

$$p_{q,r}(\alpha_q, \alpha_r) = \Pr_{X_1, X_2, \ldots}\{X_r(\{\alpha_r\})/X_q(\{\alpha_q\})\}$$
$$= \frac{\Pr_{X_1, X_2, \ldots}\{X_q(\{\alpha_q\}), X_r(\{\alpha_r\})\}}{\Pr_{X_1, X_2, \ldots}\{X_q(\{\alpha_q\})\}}. \tag{3.9}$$

The following relations can be easily deduced:

$$p_r(\alpha_r) = \sum_{\alpha_q=1}^{n} p_{q,r}(\alpha_q, \alpha_r)\,p_q(\alpha_q), \tag{3.10}$$

$$\sum_{\alpha_r=1}^{n} p_{q,r}(\alpha_q, \alpha_r)\,p_{r,s}(\alpha_r, \alpha_s) = p_{q,s}(\alpha_q, \alpha_s), \tag{3.11}$$

for all distinct q, r, $s \geq 1$ and α_q, α_r, $\alpha_s \in \{1, 2, \ldots, n\}$. The relation (3.11) is the famous Smoluchowski—Chapman—Kolmogorov (SCK) relation which is of fundamental importance in the theory of classical Markov chains also.[23]

It is important to realize that the relations (3.10) and (3.11) hold only between the operationally meaningful singlet and transition probabilities. If we had proceeded just by analogy with classical probability theory, we would be led to consider quantities like $\omega_{q,s}(\alpha_q, \alpha_s)$ given by

$$\omega_{q,s}(\alpha_q, \alpha_s) = \mu(X_q(\{\alpha_q\}) \wedge X_s(\{\alpha_s\}))/\mu(X_q(\{\alpha_q\})). \tag{3.12}$$

$\omega_{q,s}$ is meaningful as a transition probability of the Markov chain[24] $\{X_1, X_2, \ldots, X_r, \cdots\}$ only when $q = s - 1 = 1$.

In conclusion, we may note that our considerations clearly show that in general the various joint and conditional probabilities in quantum theory, which refer to situations where *different* sequences of experiments are performed on the system, *are not* related to each other as in classical probability theory. However, relations such as (3.10), (3.11) illustrate also the following important property of the quantum theoretic probabilities [which can be easily established on the basis of Eq. (2.5), (2.7)]: *The various joint and conditional probabilities, all of which refer to the same situation (i.e., the same sequence of experiments being performed on the system), satisfy among themselves all the relations of classical probability theory.*

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Entropy and phase transitions in partially ordered sets

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We define the entropy function $S(\rho) = \lim_{n\to\infty} 2n^{-2}\ln N(n,\rho)$, where $N(n,\rho)$ is the number of different partial order relations definable over a set of n distinct objects, such that of the possible $n(n-1)/2$ pairs of objects, a fraction ρ are comparable. Using rigorous upper and lower bounds for $S(\rho)$, we show that there exist real numbers ρ_1 and ρ_2; $.083 < \rho_1 \le 1/4$ and $3/8 \le \rho_2 < 48/49$; such that $S(\rho)$ has a constant value $(\ln 2)/2$ in the interval $\rho_1 \le \rho \le \rho_2$; but is strictly less than $(\ln 2)/2$ if $\rho \le .083$ or if $\rho \ge 48/49$. We point out that the function $S(\rho)$ may be considered to be the entropy function of an interacting "lattice gas" with long-range three-body interaction, in which case, the lattice gas undergoes a first order phase transition as a function of the "chemical activity" of the gas molecules, the value of the chemical activity at the phase transition being 1. A variational calculation suggests that the system undergoes an infinite number of first order phase transitions at larger values of the chemical activity. We conjecture that our best lower bound to $S(\rho)$ gives the exact value of $S(\rho)$ for all ρ.

In this paper we discuss the asymptotic enumeration of partial order relations defined over a set of n distinct objects when a finite fraction ρ of the $n(n-1)/2$ pairs are comparable.

Let $N(n)$ be the total number of partial order relations defined over n objects. It is easy to show that

$$N(n) \geq 2^{n^2/4}. \tag{1}$$

Kleitman and Rothschild[1] have shown that

$$\ln N(n) \leq \frac{n^2}{4} \ln 2 + A n^{3/2} \ln n, \tag{2}$$

for some finite constant A. Combining these two results we see that

$$\lim_{n\to\infty} \frac{2}{n^2} \ln N(n) = \tfrac{1}{2} \ln 2. \tag{3}$$

We are here interested in a more detailed asymptotic enumeration of partial order relations. For this purpose, we define the function

$$S(\rho) \stackrel{\text{def}}{=} \lim_{n\to\infty} 2n^{-2} \ln N(n,\rho), \tag{4}$$

where $N(n,\rho)$ is the number of partial order relations on n distinct objects such that $\rho n(n-1)/2$ of the $n(n-1)/2$ pairs are comparable. [We call a pair (a,b) comparable if $a > b$ or $b > a$.] If $\rho n(n-1)/2$ is not integral, we round it off to the nearest integer. The difference is clearly unimportant for large n. Clearly we have

$$S(0) = S(1) = 0, \tag{5}$$

and from (2)

$$S(\rho) \leq \tfrac{1}{2} \ln 2. \tag{6}$$

Our first results about $S(\rho)$ are given in Theorem 1.

Theorem 1: (i) $S(\rho)/\rho$ is a monotonic nonincreasing function of ρ.

(ii) $S(\rho)/(1-\rho)$ is a monotonic nondecreasing function of ρ.

Proof: (i) From any partial order relation on n objects, we can generate a partial order relation on $(n+\epsilon)$ objects by introducing ϵ new elements, incomparable to all of the n elements and to each other. We thus have the trivial inequality

$$N(n,\rho) \leq N\left(n+\epsilon, \frac{\rho n(n-1)}{(n+\epsilon)(n+\epsilon-1)}\right). \tag{7}$$

The theorem follows if we take the logarithms of both sides and the limit of n and ϵ going to infinity.

(ii) The proof is similar to that of (i). Add of a chain of ϵ new elements to the original set of n elements such that any of the new ϵ elements is less than any of the original n elements. The density of comparable pairs in this new set of $(n+\epsilon)$ elements is

$$\rho' = \frac{2}{(n+\epsilon)(n+\epsilon-1)} \left(\frac{n(n-1)}{2}\rho + n\epsilon + \frac{\epsilon(\epsilon-1)}{2}\right). \tag{8}$$

The result follows from the inequality

$$N(n+\epsilon, \rho') \geq N(n,\rho), \tag{9}$$

by taking the logarithms of both sides and going to the limit of large n and ϵ. ■

We note that Theorem 1 implies that $S(\rho)$ is a continuous function of ρ. It is quite likely that the results of this theorem can be made stronger. In particular, we would like to prove that $S(\rho)$ is a convex function of ρ. At the present time, however, the convexity of $S(\rho)$ is an unproved conjecture. We now derive a lower bound for $S(\rho)$.

Theorem 2: Let f_i, p be any positive real numbers satisfying the following conditions:

(i) $f_i \geq 0$,

(ii) $0 \leq p \leq 1$,

(iii) $\sum_i f_i = 1$,

(iv) $\sum_i [f_i^2 + 2 f_i f_{i+1} (1-p)] = 1 - \rho$.

Then

$$S(\rho) \geq 2\left(\sum_i f_i f_{i+1}\right)[-p \ln p - (1-p)\ln(1-p)],$$

Proof: We consider a set of n distinct elements where n is very large. This may be divided into disjoint subsets so that the ith subset contains nf_i elements. (For simplicity, we assume that nf_i are all integers. This is clearly inessential as we let n tend to infinity in the end.) We now construct a partial order relation amongst these objects as follows:

1. Any element in the ith subset is greater than any element in the jth subset if $j > i+1$.

2. Elements belonging to the same subset are noncomparable.

3. In no case is an element in the $(i+1)$th subset greater than an element in the ith subset.

Any relation which satisfies these conditions is a partial order relation. To complete the construction, we have to specify the relation between the $n^2 (\sum_i f_i f_{i+1}) = N_1$ (say) pairs of the type (a,b), where a and b belong to the ith and $(i+1)$th subsets repectively for some i. We arbitrarily set $a > b$ for pN_1 of these pairs and a incomparable to b for the rest. The resulting relation has a fraction ρ of all the pairs comparable.

Total no. of such relations $= {}^{N_1}C_{pN_1} \leq N(n,\rho)$.
Taking the logarithms and going to the limit of large n, we get Theorem 2.

Corollary 2.1: If $\frac{1}{4} \leq \rho \leq \frac{3}{8}$, then $S(\rho) = \frac{1}{2} \ln 2$. ■

Proof: Choose

$$f_1 = \tfrac{1}{4} + \left(\frac{3}{16} - \frac{\rho}{2}\right)^{1/2}, \quad f_2 = \tfrac{1}{2}, \quad f_3 = \tfrac{1}{4} - \left(\frac{3}{16} - \frac{\rho}{2}\right)^{1/2}, \quad p = \tfrac{1}{2}.$$

Then Theorem 2 gives us $S(\rho) \geq \frac{1}{2} \ln 2$.
Combined with Eq. (6), this proves the corollary. ■

Corollary 2.2: If $\rho \leq \frac{1}{4}$; then $S(\rho) \geq \frac{1}{2}[-2\rho \ln(2\rho) - (1-2\rho)\ln(1-2\rho)]$.

Proof: Put $f_1 = f_2 = \frac{1}{2}$, $p = 2\rho$ in Theorem 2. ■

Corollary 2.3: If $\rho > \dfrac{3}{8}$, then $S(\rho) \geq \dfrac{4}{5}(1-\rho)\ln 2$.

Proof: This follows from Theorem 1 by putting $S(\frac{3}{8}) = \frac{1}{2} \ln 2$. ■

We can determine better lower bounds for $S(\rho)$ than given by Corollary 2.3 by using Theorem 2 and variational calculus to choose f_i so that the largest lower bound is attained. Using Lagrange's multipliers, it is easy to show that the optional choice of $\{f_i, p\} = \{f_i^*, p^*\}$ satisfies the conditions

$$f_i^*\left(\frac{\ln(1-p^*)}{\ln p^*} - 1\right) - f_{i-1}^* - f_{i+1}^* \begin{cases} = M, & \text{if } f_i^* > 0, \\ \geq M, & \text{if } f_i^* = 0. \end{cases} \tag{10}$$

p^* and M are chosen so that the corresponding solution $\{f_i^*\}$ satisfies conditions (i)–(iv) of Theorem 2. This determines p^* uniquely for a given ρ. If however

$$\frac{\ln(1-p^*)}{\ln p^*} - 1 = 2\cos\frac{2\pi}{r}, \quad r = 4, 5, 6 \cdots; \tag{11}$$

then the corresponding solution $\{f_i^*\}$, and hence ρ, is not unique for a given value of p^*. The graph of p^*, as a function of ρ, shows intervals of ρ for which the value of p^* is a constant. It is easy to verify that in each of these intervals $S^*(\rho)$, our best lower bound to $S(\rho)$, is a linear function of ρ.

It is quite plausible that the optimal lower bound given by Theorem 2 gives us the exact value of $S(\rho)$. The only partial order relations not counted in Theorem 2 are those containing at least one incomparable (a,b) such that a and b belong to the ith and jth subsets respectively with $j > i+1$, for some i,j. This however implies that no element of the $(i+1)$th subset is simultaneously comparable to both a and b; and the probability of such an event tends to zero exponentially for large n. We conjecture that the best lower bound to $S(\rho)$ given by Theorem 2, coincides with the exact value of $S(\rho)$ for all ρ.

We now obtain upper bounds for $S(\rho)$, which are stronger than (6) in some interval of ρ.

If ρ is very small, it is easy to see that

$$S(\rho) \leq \rho \ln 2 - \rho \ln \rho - (1-\rho)\ln(1-\rho). \tag{12}$$

To prove this, we just observe that there are ${}^{n(n-1)/2}C_{n(n-1)\rho/2}$ ways of choosing $\rho n(n-1)/2$ comparable pairs out of $n(n-1)/2$, and there are at most two possibilities of ordering for each comparable pair. Taking the logarithms and limit of large n gives us (12). In particular, we note that (12) implies that

$$S(\rho) < \tfrac{1}{2}\ln 2 \text{ if } \rho < .083. \tag{13}$$

While (12) gives a fairly good upper bound if ρ is very small, it is quite worthless if ρ is close to 1 and $(1-\rho)$ is small. In this case a better upper bound is given by the following theorem:

Theorem 3: $S(\rho) \leq 4 \ln 2\, (1-\rho)^{1/2}$.

Proof: We note that the maximum number of mutually noncomparable objects in a partial order relation on n objects with $\rho n(n-1)/2$ comparable pairs is less than $n(1-\rho)^{1/2} + 1 = m$ (say). Hence, by Dilworth's theorem,[2] we can choose m chains such that their union contains all the n elements.

Let the lengths of these chains be $l_1, l_2, l_3, \ldots, l_m$ in decreasing order of magnitude. Consider now any two chains ith and jth. Let N_{ij} be the number of different ways we may assign a partial order relation on the set formed by the union of these chains consistent with their chain structure. Then we have

$$N_{ij} \leq [(l_i + l_j)!/l_i!\,l_j!]^2. \tag{14}$$

This may be seen as follows: Let the elements of the ith chain be $a_1 > a_2 > a_3 \cdots > a_{l_i}$, and the elements of the jth chain be $b_1 > b_2 > \cdots b_{l_j}$. Then a partial order relation over the combined set of $(l_i + l_j)$ elements is uniquely specified by a list of $(l_i + 2l_j)$ elements. a_α, b'_β and b''_β $(\alpha = 1$ to l_i, $\beta = 1$ to $l_j)$ of the type

$$a_1 a_2 b'_1 a_3 b'_2\, a_4 a_5 b'_3 b''_1 \cdots.$$

In this list b'_β occurs after all the elements of the ith chain which are greater than b_β, and before all the elements of the ith chain which are not. Similarly b''_β occurs after all elements of the ith chain which are not less than b_β, and before all elements that are. Clearly if $\beta_1 > \beta_2$, then b'_{β_1} occurs after b'_{β_2} in the list, and b''_{β_2} occurs after b''_{β_2}. We further assume that if in this list, there is an uninterrupted string of b's, then all b''_β's occur after b'_β's in that string.

The number of ways we may insert a chain of b'_β's in the chain of a's is $(l_i + l_j)!/(l_i!\,l_j!)$. Similarly for b''_β's. Hence the total number of such lists is equal to

$$[(l_i+l_j)!/l_i!l_j!]^2 \qquad (15)$$

Not all these lists correspond to partial order relations. In particular, b'_β must precede b''_β in the list for all β, for the list to correspond to a partial order relation. This proves (14).

Now, the number of ways m disjoint chains may be chosen out of n elements is ${}^{n+m-1}P_n$. Hence the total number of partial order relations having at most m noncomparable elements is

$$\leq {}^{n+m-1}P_n \max_{\{l_i\}} \left[\prod_{\substack{i,j=1\\ i<j}}^{m} \frac{(l_i+l_j)!}{l_i!\, l_j!} \right]^2 \qquad (16)$$

where the maximum value of the term inside the square brackets is to be taken over all m partitions of n (i.e., $\sum_{i=1}^m l_i = n$). The maximum is attained if all l_i are equal to n/m. We drop here the constraint of l_i being integers. Taking the logarithm of the resulting inequality and retaining only the terms of order n^2, we get

$$\frac{n^2}{2}\ S(\rho) \leq \frac{n^2}{2}\ (1-\rho)^{1/2}\ 4 \ln 2 + O(n^2). \qquad (17)$$

Taking the limit $n \to \infty$, we obtain

$$S(\rho) \leq (1-\rho)^{1/2}\ 4 \ln 2. \blacksquare \qquad (18)$$

Corollary 3.1: If $\rho > 48/49$, then, $S(\rho) < (\ln 2)/2$.

Proof: From Theorem 3, $S(48/49) \leq (2/49) \ln [14!/(7!)^2] < \frac{1}{2} \ln 2$. ■

Theorem 3 is not the best possible. Of all the possible decomposition of a partial order relation into chains; we may choose the one which gives the largest value of $\sum_{i=1}^m l_i^2$. (Some of the l_i's may be zero.) Then each of the elements of the ith chain is incomparable to at least one element in each of the preceding chains $j<i$. This give the inequality

$$\sum_{i=1}^{m} (i-1)\ l_i \leq (1-\rho) n(n-1)/2. \qquad (19)$$

This constraint, in addition to sharper bounds on N_{ij}, may be used to obtain an improved upper bound to $S(\rho)$. These bounds are, however, still far above the true value of $S(\rho)$. In any case, Theorem 3 is quite sufficient to prove that $S(\rho)$ is nonanalytic.

Putting together the results of this paper, we see that $S(\rho)$ is a continuous function of ρ in the allowed range of variation of ρ, $0 \leq \rho \leq 1$. It, however, is a nonanalytic function of ρ, and there exist numbers ρ_1 and ρ_2 such that

$$S(\rho) = \tfrac{1}{2} \ln 2 \ \text{ for } \rho_1 \leq \rho \leq \rho_2.$$

We have shown that $S(\rho)$ is strictly less than $\frac{1}{2}\ln 2$, if $\rho \leq .083$ or if $\rho \geq 48/49$. Hence we get from Corollary 2.1

$$.083 \leq \rho_1 < \tfrac{1}{4}, \qquad (20)$$

and

$$\tfrac{3}{8} \leq \rho_2 < 48/49. \qquad (21)$$

We may interpret $S(\rho)$ to be the entropy per particle of an interacting "lattice gas." Here the "lattice sites" are the $n(n-1)/2$ pairs of elements. The three possible states ($a>b$, $a<b$ or a $\not R b$) of a pair (a,b) under a reflexive antisymmetric binary relation correspond to three possible "states" of a lattice site in the interacting gas language. A "configuration" of the 'gas' corresponds to a reflexive, antisymmetric binary relation on n elements, and is specified by specifying the "state" of eqch "lattice site." We call the $(a \not R b)$ state the "unoccupied" state of the lattice site (a,b). $a>b$ and $a<b$ correspond to two different possible states of the gas molecule at the "occupied site" (a,b).

The transititivity property of the partial order relations corresponds to a 3-body interaction between lattice sites. The interaction is hard-core type, in the sense that it excludes certain configurations from the statistical sum, or alternatively puts their weight equal to zero. This condition may be relaxed and the properties of soft core systems may be of interest.

The flat portion of the $S(\rho)$ curve corresponds to a first order phase transition in the interacting lattice gas. The corresponding value of the chemical potential of the lattice gas is zero $\mu = -\partial S/\partial\rho$. This corresponds to the chemical activity of the gas being 1. We may speak of the states with $\rho < \rho_1$ and $\rho > \rho_2$ constituting the 'disordered phase ' and the 'ordered phase' respectively, In the language of partial order relations, the ordered phase is characterized by a larger value of "average maximal chain length."

The greatest lowerbound approximation to $S(\rho)$, as given by Theorem 2, shows that $S(\rho)$, as a function of ρ, contains an infinite number of linear segments. In the language of phase transitions, the system exhibits an infinite number of first order phase transitions. The different phases corresponds to different values of "average maximal chain length," which serves the role of the order parameter in this system. The order parameter jumps by one unit across a phase transition.

The relationship of these phase transitions to phase transitions in realistic physical systems, if any, is not very clear. The transitions are governed by the strong, long range nature of the 3-body interaction here, not usually encountered in physical systems. While the asymptotic enumeration of partial order relations is of sufficient interest intrinsically, the study of the mechanism of these transitions may be of some interest in statistical physics. In particular, the distribution of zeros of the grand partition function[3,4] of this system may be of some interest.

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On the higher approximations of the K-harmonics method

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Exact formulas for the calculation of two-body central interactions in the K-harmonics method are derived. These formulas hold for any approximations and account for diagonal and off-diagonal matrix elements as well.

1. INTRODUCTION

The K-harmonics method of Simonov and Badalyan[1–4] have proved to be a very convenient approach to the problem of finding the energy levels and the corresponding wavefunctions of the bound states of the nuclear many-body problem with two-body interactions.

An approximation scheme was set in the method by expanding the wavefunctions of the bound states of the system, in the c.m. frame, in terms of angular functions which form a complete set of functions in the unit sphere of $E_{3(A-1)}$, the vector space spanned by the relative vectors of the A particles which constitute the nuclear system. These angular functions also contain the spin and isospin coordinates of the nucleons and are referred to as K-harmonics when they are totally antisymmetric and constitute the angular part of harmonic and homogeneous polynomials of degree K in the spatial relative variables. The successive approximations of the method are obtained by considering, in the expansion of the wavefunctions and in the matrix elements of the two-body interactions, only the K-harmonics with K up to $K_{\min}$, $K_{\min}+1, \circ\circ\circ$, where $K_{\min}$ is the minimum value, compatible with the Pauli principle, that K can assume for the A-particle system under study.

The K-harmonics method has been applied with great success to light nuclei such as ^{3}H, ^{3}He, and ^{4}He (Refs. 1–10) to all orders of approximation with an excellent convergence rate. It was also applied to less light nuclei such as ^{10}He, $^{15-17}$O, and ^{40}Ca (Refs. 11–13) in first approximation using the approximate formulas of Baz and Zukov[14] and more recently to ^{16}O (Ref. 15), using the exact first-approximation formulas of Gorbatov.[16]

In order to study the convergence of the method for any nuclei it is necessary to derive formulas which give the scalar products as well as the matrix elements of the two-body interactions for any K-harmonic. The difficulty in obtaining these formulas is the fact that the integrations have to be performed in $S_{3(A-1)}$, the unit sphere of $E_{3(A-1)}$ instead of $E_{3(A-1)}$. Despite this difficulty some progress was made in the program of obtaining these formulas. The first step in this program was given by Baz and Zukov[14] who derived approximate formulas for the first approximation. Further steps were given by Gorbatov[16] and Castilho Alcaras[17] who derived exact first-order approximation formulas for diagonal and off-diagonal matrix elements respectively.

In this paper we complete this program deriving exact formulas for any K-harmonics. We shall use the same techniques as Ref. 17, from which we borrow, whenever possible, the notation.

In Sec. 2 we introduce what we call the K-angular functions and show that working with these functions is equivalent to working with the K-harmonics. In Sec. 3 we restate Gorbatov's theorem[16] and use it to derive the scalar products of two K-angular functions and the matrix elements of the two-body operators between such functions. Finally, in Appendices A and B, we derive some results that are needed in Sec. 2.

2. THE K-ANGULAR FUNCTIONS AND THE K-HARMONICS

Antisymmetric and homogeneous polynomials of degree K can be constructed by filling a Slater determinant with orbitals $\phi_j(i)$ whose space parts are homogeneous polynomials of degree K_j in the spatial relative coordinates $\boldsymbol{\rho}_i=\mathbf{R}-\mathbf{r}_i$ of the A nucleons such that the sum of the degrees equals K. Following Gorbatov we shall use the orbitals

$$\phi_j(i)\equiv\phi^{\mu_j\tau_j}_{a_jb_jc_j}(i)=(\rho_{xi})^{a_j}(\rho_{yi})^{b_j}(\rho_{zi})^{c_j}\Lambda_{\mu_j\tau_j}(i)\ ,$$
$$i,j=1,2,\ldots,A,\quad a_j,b_j,c_j=0,1,2,\circ\circ\circ,$$
$$\mu_j,\tau_j=+\tfrac{1}{2},-\tfrac{1}{2}, \tag{2.1}$$

where ρ_{xi}, ρ_{yi}, ρ_{zi} are the Cartesian components of $\boldsymbol{\rho}_i$ and $\Lambda_{\mu_j\tau_j}(i)$ accounts for the spin–isospin variables of particle i. We call a *K-angular function with orbital labels set* $\{\phi\}$, the following angular function,

$$U^K_A(\{\phi\})=\frac{\mathcal{N}}{\rho^K}\begin{vmatrix}\phi_1(1) & \cdots & \phi_1(A)\\ \vdots & & \vdots\\ \phi_A(1) & \cdots & \phi_A(A)\end{vmatrix}, \tag{2.2}$$

where $\mathcal{N}$ is a normalization constant,

$$K=\sum_{i=1}^{A}(a_i+b_i+c_i), \tag{2.3}$$

$$\rho=\left[\sum_{i=1}^{A}\rho_i^2\right]^{1/2}, \tag{2.4}$$

and $\{\phi\}$ denotes the set of orbital labels:

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$$\{\phi\} = \begin{array}{l} a_1, b_1, c_1, \mu_1, \tau_1, \\ a_2, b_2, c_2, \mu_2, \tau_2, \\ \text{-------------} \\ a_A, b_A, c_A, \mu_A, \tau_A. \end{array} \tag{2.5}$$

The Slater determinant in Eq. (2.2) is an antisymmetric and homogeneous polynomial of degree K. When $K=K_{\min}$ and $K_{\min}+1$ this polynomial is also harmonic[3] and the corresponding K-angular function $U_A^K(\{\phi\})$ is a K-harmonic. When $K \geq K_{\min}+2$ the Slater determinant will no longer be a harmonic polynomial and consequently $U_A^K(\{\phi\})$ will not be a K-harmonic. Even so those K-angular functions are still useful as can be seen by the arguments that follow.

In Appendix A we show that the Laplacian of $E_{3(A-1)}$ may be expressed in terms of the ρ_i as

$$\nabla^2 = \sum_{i=1}^{A} \nabla_{\rho_i}^2 - \frac{1}{A}\left(\sum_{i=1}^{A} \nabla_{\rho_i}\right) \cdot \left(\sum_{j=1}^{A} \nabla_{\rho_j}\right). \tag{2.6}$$

Observe that, written in this way, the $E_{3(A-1)}$ Laplacian becomes a function of one-body operators acting on the relative vectors. In Appendix B we show that the one-body operators acting on the relative vectors transform a Slater determinant into a linear combination of Slater determinants. One then concludes that one can easily find the effect of ∇^2 on a Slater determinant with orbitals (2.1).

Denoting by $|K-2;i\rangle$, $i=1,2,\cdots$ the linearly independent *harmonic* Slater determinants of degree $K-2$, and by $|K;j\rangle$ a Slater determinant of degree K, one has

$$\nabla^2 |K;j\rangle = \sum_i D_{ji} |K-2;i\rangle. \tag{2.7}$$

It follows from Eq. (2.7) that, in order to have

$$\nabla^2 \left(\sum_j \lambda_j |K;j\rangle\right) = 0, \tag{2.8}$$

the coefficients λ_j must satisfy the following overdetermined system of linear equations,

$$\sum_i D_{ij} \lambda_i = 0. \tag{2.9}$$

In conclusion, one finds that the K-harmonics are some suitable linear combinations of K-angular functions. Therefore, the scalar products of two K-harmonics and the matrix elements of the two-body interactions between two K-harmonics is determined if we know the corresponding scalar products and matrix elements for the K-angular functions $U_A^K(\{\phi\})$. For this reason we shall be dealing exclusively with K-angular functions instead of K-harmonics.

3. SCALAR PRODUCTS AND MATRIX ELEMENTS OF TWO-BODY CENTRAL INTERACTIONS

Following what was done in Refs. 16 and 17 for the $K_{\min}$ K-harmonic, we write the K-angular functions (2.2) in the operatorial form

$$U_A^K(\{\phi\}) = \frac{\mathcal{N}}{(i\rho)^K} D_A^K(\{\phi\})_{\mathbf{k}} \exp\left\{ i \prod_{j=1}^{A} \mathbf{k}_j \cdot \rho_j \right\}, \tag{3.1}$$

by defining the antisymmetrizer operator of degree K and orbital labels set $\{\phi\}$ for a system of A particles as

$$D_A^K(\{\phi\})_{\mathbf{k}} = \lim_{\{\mathbf{k}_i\}\to 0} \sum_P \epsilon_P \prod_{i=1}^{A} \Lambda_{\mu_{Pi}\tau_{Pi}}^{(i)} \times \left(\frac{\partial}{\partial k_{xi}}\right)^{a_{Pi}} \left(\frac{\partial}{\partial k_{yi}}\right)^{b_{Pi}} \left(\frac{\partial}{\partial k_{zi}}\right)^{c_{Pi}}. \tag{3.2}$$

In Eq. (3.2), $P \equiv (P1, P2, \ldots, PA)$ is a permutation of A objects, k_{xi}, k_{yi}, k_{zi} are the Cartesian components of $\mathbf{k}_i$ and $\{\phi\}$ is the set of orbital labels (2.5) subject to the degree condition (2.3).

Gorbatov[16] proved that for a function $\mathcal{S}(\{\mathbf{k}_i\})$ that is symmetric in $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_A$ one has[18]

$$D_A^K(\{\phi\})_{\mathbf{k}} \mathcal{S}(\{\mathbf{k}_i\}) = \prod_{i=1}^{A} \left[\sum_{\alpha_i=0}^{a_i} \sum_{\beta_i=0}^{b_i} \sum_{\gamma_i=0}^{c_i} \binom{a_i}{\alpha_i}\binom{b_i}{\beta_i}\binom{c_i}{\gamma_i} \lim_{\{\mathbf{k}_i\}\to 0} \left(\frac{\partial}{\partial k_{xi}}\right)^{\alpha_i} \times \left(\frac{\partial}{\partial k_{yi}}\right)^{\beta_i} \left(\frac{\partial}{\partial k_{zi}}\right)^{\gamma_i} \mathcal{S}(\{\mathbf{k}_i\}) \right] D_A^{\bar{K}}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})_{\mathbf{k}}, \tag{3.3}$$

where, for each one of the $(3A)$ tuples $(\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_A, \beta_A, \gamma_A)$, the orbital labels set $\{\phi(\alpha_j,\beta_j,\gamma_j)\}$ and associate degree $\bar{K} \equiv \bar{K}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})$ are given by

$$\{\phi(\alpha_j,\beta_j,\gamma_j)\} = \left\{ \begin{array}{l} a_1-\alpha_1,\ b_1-\beta_1,\ c_1-\gamma_1,\ \mu_1, \tau_1, \\ a_2-\alpha_2,\ b_2-\beta_2,\ c_2-\gamma_2,\ \mu_2, \tau_2, \\ \text{-------------------------} \\ a_A-\alpha_A, b_A-\beta_A, c_A-\gamma_A, \mu_A, \tau_A, \end{array} \right. \tag{3.4}$$

$$\bar{K} = \sum_{j=1}^{A} (a_j+b_j+c_j-\alpha_j-\beta_j-\gamma_j) \leq K. \tag{3.5}$$

When $K=K_{\min}$ Eq. (3.5) implies that only $\bar{K}=K_{\min}$ is actually present in (3.3) since all the antisymmetrizers with $K<K_{\min}$ vanish identically. Then in that case one has

$$D_A^{K_{\min}}(\{\phi\})_{\mathbf{k}} \mathcal{S}(\{\mathbf{k}_i\}) = \mathcal{S}(\{\mathbf{k}_i=0\}) D_A^{K_{\min}}(\{\phi\})_{\mathbf{k}}, \tag{3.6}$$

for any symmetric function $\mathcal{S}(\{\mathbf{k}_i\})$.

Let us consider now the scalar product of two K-angular functions

$$\langle U_A^{K'}(\{\phi'\}) | U_A^K(\{\phi\})\rangle = \int U_A^{K'}(\{\phi'\})^\dagger U_A^K(\{\phi\})\, d\Omega_{3(A-1)} \tag{3.7}$$

and the matrix elements

$$\langle U_A^{K'}(\{\phi'\}) | \hat{F}_2 | U_A^K(\{\phi\})\rangle = \int U_A^{K'}(\{\phi'\})^\dagger \hat{F}_2 U_A^K(\{\phi\})\, d\Omega_{3(A-1)} \tag{3.8}$$

of the two-body central operator

$$\hat{F}_2 = \sum_{i>j=1}^{A} v(|\rho_i - \rho_j|) \mathcal{O}(i,j), \tag{3.9}$$

where $\mathcal{O}(i,j)$ is an operator which acts only on the spin–isospin variables of the pair of particles (i,j).

The rhs's of Eqs. (3.7) and (3.8) are evaluated by using Eqs. (3.1)–(3.3) and Eq. (16) of Ref. 17. Since this program was already carried out in some detail in Ref. 17 for $K=K_{\min}$ by use of Eq. (3.6) instead of Eq. (3.3), it is unnecessary to repeat that material; it suffices to examine the effects of replacing the mentioned equations.

In Ref. 17, Eq. (3.6) was used to replace

$$F_K^{K'}(\{\phi\},\{\phi'\},t) = \int d\mathbf{f} \exp\left(-\frac{Aif^2}{4t}\right) D_A^{K'}(\{\phi'\})_{\mathbf{k}'}^{\dagger} D_A^K(\{\phi\})_{\mathbf{k}} \mathcal{S}(\{\mathbf{k}_i\},\{\mathbf{k}_i'\},\mathbf{f},t) \tag{3.10}$$

by

$$F_K^{K'}(\{\phi\},\{\phi'\},t) = \int d\mathbf{f} \exp\left(-\frac{Aif^2}{4t}\right) D_A^{K'}(\{\phi'\})_{\mathbf{k}'}^{\dagger} D_A^K(\{\phi\})_{\mathbf{k}} \tag{3.11}$$

since

$$\mathcal{S}(\{\mathbf{k}_i\},\{\mathbf{k}_i'\},\mathbf{f},t) = \exp\left(-\frac{i}{4t}\sum_{s=1}^{A}[(\mathbf{k}_s^2+\mathbf{k}_s'^2)+2\mathbf{f}\cdot(\mathbf{k}_s-\mathbf{k}_s')]\right) \tag{3.12}$$

is a symmetric function of both $\{\mathbf{k}_i\}$ and $\{\mathbf{k}_i'\}$ which goes to one when $\{\mathbf{k}_i\}$ and $\{\mathbf{k}_i'\}$ go to zero and in that case one had $K=K'=K_{\min}$.

Now, instead of (3.11) one has, using (3.3) twice,

$$\begin{aligned} F_K^{K'}(\{\phi\},\{\phi'\},t) &= \prod_{l=1}^{A}\sum_{\alpha_l=0}^{a_l}\sum_{\beta_l=0}^{b_l}\sum_{\gamma_l=0}^{c_l}\sum_{\alpha_l'=0}^{a_l'}\sum_{\beta_l'=0}^{b_l'}\sum_{\gamma_l'=0}^{c_l'}\binom{a_l}{\alpha_l}\binom{b_l}{\beta_l}\binom{c_l}{\gamma_l} \\ &\times\binom{a_l'}{\alpha_l'}\binom{b_l'}{\beta_l'}\binom{c_l'}{\gamma_l'}\Bigg[\lim_{\{\mathbf{k}_i\},\{\mathbf{k}_i'\}\to 0}\left(\frac{\partial}{\partial k_{xl}}\right)^{\alpha_l}\left(\frac{\partial}{\partial k_{yl}}\right)^{\beta_l} \\ &\times\left(\frac{\partial}{\partial k_{zl}}\right)^{\gamma_l}\left(\frac{\partial}{\partial k_{xl}'}\right)^{\alpha_l'}\left(\frac{\partial}{\partial k_{yl}'}\right)^{\beta_l'}\left(\frac{\partial}{\partial k_{zl}'}\right)^{\gamma_l'} \\ &\times\int d\mathbf{f}\exp\left(-\frac{if^2}{4t}\right)\mathcal{S}(\{\mathbf{k}_i\},\{\mathbf{k}_i'\},\mathbf{f},t)\Bigg] \\ &\times D_A^{\bar{K}'}(\{\phi'(\alpha_j',\beta_j',\gamma_j')\})_{\mathbf{k}'}^{\dagger}\, D_A^{\bar{K}}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})_{\mathbf{k}}\ . \end{aligned} \tag{3.13}$$

From Eq. (3.12) one has

$$\begin{aligned} &\lim_{k_{xl}\to 0}\left(\frac{\partial}{\partial k_{xl}}\right)^n \mathcal{S}(\{\mathbf{k}_i\},\{\mathbf{k}_i'\},\mathbf{f},t) \\ &= \exp\left(\frac{i}{4t}(k_{xl}^2+2f_xk_{xl})\right)\mathcal{S}\lim_{k_{xl}\to 0}\left(\frac{\partial}{\partial k_{xl}}\right)^n \\ &\quad\times\exp\left(-\frac{i}{4t}(k_{xl}^2+2f_xk_{xl})\right) = \exp\left(\frac{i}{4t}(k_{xl}^2+2f_xk_{xl})\right) \\ &\quad\times\mathcal{S}(-)^n\left(\frac{i}{4t}\right)^{n/2}H_n\left(\left(\frac{i}{4t}\right)^{1/2}f_x\right),\quad n=0,1,2,\cdots, \end{aligned}$$

where in the last step use was made of the generating function of the Hermite polynomials,[19] $H_n(x)$.

From Eq. (3.14) it follows that the quantity between brackets in Eq. (3.13) is equal to

$$\begin{aligned} &(-)^{K-\bar{K}}\left(\frac{i}{4t}\right)^{(K-\bar{K}+K'-K'-\bar{K}')/2}\int d\mathbf{f}\exp\left(-\frac{Aif^2}{4t}\right) \\ &\times\left[\prod_{l=1}^{A}H_{\alpha_l}\left(\left(\frac{i}{4t}\right)^{1/2}f_x\right)H_{\alpha_l'}\left(\left(\frac{i}{4t}\right)^{1/2}f_x\right)H_{\beta_l}\left(\left(\frac{i}{4t}\right)^{1/2}f_y\right)\right. \\ &\left.\times H_{\beta_l'}\left(\left(\frac{i}{4t}\right)^{1/2}f_y\right)H_{\gamma_l}\left(\left(\frac{i}{4t}\right)^{1/2}f_z\right)H_{\gamma_l'}\left(\left(\frac{i}{4t}\right)^{1/2}f_z\right)\right]. \end{aligned} \tag{3.15}$$

Now, for each component f_j of $\mathbf{f}$ one expands the product of the $2A$ Hermite polynomials which depend on it in powers of $\sqrt{i/4t}\,f_j$. Since each Hermite polynomial has a definite parity, the resulting polynomial in f_j will also have a definite parity given by the sum of the degrees of the $2A$ Hermite polynomials involved and therefore the integration in f_j will vanish when such parity is odd. Performing the integrations in f_x, f_y, f_z one then obtains

$$\begin{aligned} F_K^{K'}(\{\phi\},\{\phi'\},t) &= \prod_{l=1}^{A}\Bigg[\sum_{\alpha_l=0}^{a_l}\sum_{\beta_l=0}^{b_l}\sum_{\gamma_l=0}^{c_l}\sum_{\alpha_l'=0}^{a_l'}\sum_{\beta_l'=0}^{b_l'}\sum_{\gamma_l'=0}^{c_l'} \\ &\times\binom{a_l}{\alpha_l}\binom{b_l}{\beta_l}\binom{c_l}{\gamma_l}\binom{a_l'}{\alpha_l'}\binom{b_l'}{\beta_l'}\binom{c_l'}{\gamma_l'} \\ &\times B_A(\{\alpha_j\},\{\alpha_j'\})\,B_A(\{\beta_j\},\{\beta_j'\})B_A(\{\gamma_j\},\{\gamma_j'\}) \\ &\times\left(\frac{i}{4t}\right)^{(K-\bar{K}+K'-\bar{K}')/2}\left(\frac{\pi}{A}\frac{4t}{i}\right)^{3/2} \\ &\times D_A^{\bar{K}'}(\{\phi'(\alpha_j',\beta_j',\gamma_j')\})_{\mathbf{k}'}^{\dagger}D_A^{\bar{K}}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})_{\mathbf{k}}, \end{aligned} \tag{3.16}$$

where the $B_A(\{n_i\},\{n_i'\})$'s are defined through the following four steps:

(i) $\alpha=\sum_{l=1}^{A}(n_l+n_l'),\ n_l, n_l' =$ nonnegative integers; (3.17)

(ii) for $\alpha=$ odd, $B_A(\{n_i\},\{n_i'\})\equiv 0$; (3.18)

(iii) for $\alpha=$ even, define $C_m^{(A)}(\{n_i\},\{n_i'\})$ by

$$\prod_{l=1}^{A}H_{n_l}(x)\ H_{n_l'}(x)=\sum_{m=0}^{\alpha/2}C_m^{(A)}(\{n_i\},\{n_i'\})x^{\alpha-2m}; \tag{3.19}$$

(iv) for $\alpha=$ even

$$B_A(\{n_i\},\{n_i'\})=\sum_{m=0}^{\alpha/2}C_m^{(A)}(\{n_i\},\{n_i'\})\frac{(\alpha-2m)!}{(4A)^{\alpha/2-m}(\alpha/2-m!)}. \tag{3.20}$$

[In Table I we list the $B_A(\{n_i\},\{n_i'\})$ necessary when $K+K'\leq 2K_{\min}+8$.]

From now on, one follows the same steps of Ref. 17, taking into account that the integrand in t has now an extra factor

$$\left(\frac{i}{4t}\right)^{(K-\bar{K}+K'-\bar{K}')/2}$$

and the antisymmetrizers are now $D_A^{\bar{K}'}(\{\phi'(\alpha_j',\beta_j',\gamma_j')\})_{\mathbf{k}'}$ and $D_A^{K}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})_{\mathbf{k}}$.

In this way, one obtains for (3.7) the expression

$$\begin{aligned} &\langle U_A^{K'}(\{\phi'\})\,|\,U_A^K(\{\phi\})\rangle \\ &= \frac{\pi^{3(A-1)/2}\mathcal{N}\mathcal{N}'A!\,i^{K'-K}}{i^{K-K'}2^{K-K'+1}\Gamma((K+K'+3A-3)/2)}\left(\prod_{i=1}^{A}a_i'!\,b_i'!\,c_i'!\right) \\ &\times\prod_{l=1}^{A}\left[\sum_{\alpha_l=0}^{a_l}\sum_{\beta_l=0}^{b_l}\sum_{\gamma_l=0}^{c_l}\sum_{\alpha_l'=0}^{a_l'}\sum_{\beta_l'=0}^{b_l'}\sum_{\gamma_l'=0}^{c_l'}\frac{\binom{a_l}{\alpha_l}\binom{b_l}{\beta_l}\binom{c_l}{\gamma_l}}{\alpha_l'!\,\beta_l'!\,\gamma_l'!}\right] \\ &\times(-)^{K-\bar{K}}\,2^{\bar{K}}B_A(\{\alpha_i\},\{\alpha_i'\})\,B_A(\{\beta_i\},\{\beta_i'\})\,B_A(\{\gamma_i\},\{\gamma_i'\}) \\ &\times\delta_{\{\phi(\alpha_j,\beta_j,\gamma_j)\}\{\phi'(\alpha_j',\beta_j',\gamma_j')\}}, \end{aligned} \tag{3.21}$$

where

TABLE I. Values of $B_A(\{n_i\},\{n_i'\})$. It follows from Eqs. (3.17) that, for a given α, the B's depend on $\{n_i\}$ and $\{n_i'\}$ only through a partition of α into $2A$ elements (not necessarily distinct). The notation $\dot{0}$ in column 2 means that the number 0 must be repeated as many times as necessary.

α	Partition	B
0	$\{\dot{0}\}$	1
2	$\{2\dot{0}\}$	$-2+2/A$
	$\{11\dot{0}\}$	$2/A$
4	$\{4\dot{0}\}$	$12-24/A+12/A^2$
	$\{31\dot{0}\}$	$-12/A+12/A^2$
	$\{22\dot{0}\}$	$4-8/A+12/A^2$
	$\{211\dot{0}\}$	$-4/A+12/A^2$
	$\{1111\dot{0}\}$	$12/A^2$
6	$\{6\dot{0}\}$	$-120+360/A-360/A^2+120/A^3$
	$\{51\dot{0}\}$	$120/A-240/A^2+120/A^3$
	$\{42\dot{0}\}$	$-24+72/A-168/A^2+120/A^3$
	$\{411\dot{0}\}$	$24/A-144/A^2+120/A^3$
	$\{33\dot{0}\}$	$72/A-144/A^2+120/A^3$
	$\{321\dot{0}\}$	$24/A-96/A^2+120/A^3$
	$\{3111\dot{0}\}$	$-72/A^2+120/A^3$
	$\{222\dot{0}\}$	$-8+24/A-72/A^2+120/A^3$
	$\{2211\dot{0}\}$	$8/A-48/A^2+120/A^3$
	$\{21111\dot{0}\}$	$-24/A^2+120/A^3$
	$\{111111\dot{0}\}$	$120/A^3$
8	$\{8\dot{0}\}$	$1680-6720/A+10080/A^2-6720/A^3+1680/A^4$
	$\{71\dot{0}\}$	$-1680/A+5040/A^2-5040/A^3+1680/A^4$
	$\{62\dot{0}\}$	$240-960/A+2880/A^2-3840/A^3+1680/A^4$
	$\{611\dot{0}\}$	$-240/A+2160/A^2-3600/A^3+1680/A^4$
	$\{53\dot{0}\}$	$-720/A+2160/A^2-3120/A^3+1680/A^4$
	$\{521\dot{0}\}$	$-240/A+1200/A^2-2640/A^3+1680/A^4$
	$\{5111\dot{0}\}$	$720/A^2-2400/A^3+1680/A^4$
	$\{44\dot{0}\}$	$144-576/A+2016/A^2-2800/A^3+1680/A^4$
	$\{431\dot{0}\}$	$-144/A+1008/A^2-2160/A^3+1680/A^4$
	$\{422\dot{0}\}$	$48-192/A+768/A^2-1920/A^3+1680/A^4$
	$\{4211\dot{0}\}$	$-48/A+432/A^2-1680/A^3+1680/A^4$
	$\{41111\dot{0}\}$	$144/A^2-1440/A^3+1680/A^4$
	$\{332\dot{0}\}$	$-144/A+720/A^2-1680/A^3+1680/A^4$
	$\{3311\dot{0}\}$	$432/A^2-1440/A^3+1680/A^4$
	$\{3221\dot{0}\}$	$-48/A+336/A^2-1200/A^3+1680/A^4$
	$\{32111\dot{0}\}$	$144/A^2-960/A^3+1680/A^4$
	$\{311111\dot{0}\}$	$-720/A^3+1680/A^4$
	$\{2222\dot{0}\}$	$16-64/A+288/A^2-960/A^3+1680/A^4$
	$\{22211\dot{0}\}$	$-16/A+144/A^2-720/A^3+1680/A^4$
	$\{221111\dot{0}\}$	$48/A^2-480/A^3+1680/A^4$
	$\{2111111\dot{0}\}$	$-240/A^3+1680/A^4$
	$\{11111111\dot{0}\}$	$1680/A^4$

$$\delta_{\{\phi(\alpha_j,\beta_j,\gamma_j)\}\{\phi'(\alpha_j',\beta_j',\gamma_j')\}}=\begin{cases}(-)^P\\ 0\end{cases}, \qquad (3.22)$$

the first value being attained when $D_A^{\bar{K}'}(\{\phi'(\alpha_j',\beta_j',\gamma_j')\})_{\mathbf{k}'}$ and $D_A^{\bar{K}}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})_{\mathbf{k}}$ have no repeated orbitals and their sets of orbital labels are the same up to a permutation P of the orbitals. The zero value is attained if any one of the above conditions is not fulfilled. It follows from the definition of the B's and from Eq. (3.22) that K-angular functions whose K values have opposite parities are orthogonal. This is an expected result since the parity of $U_A^K(\{\phi\})$ is $(-)^K$.

For the matrix elements (3.8) one obtains the expression

$$\langle U_A^{K'}(\{\phi'\})|\hat{F}_2|U_A^K(\{\phi\})\rangle = \prod_{l=1}^{A}\Bigg[\sum_{\alpha_l=0}^{a_l}\sum_{\beta_l=0}^{b_l}\sum_{\gamma_l=0}^{c_l}\sum_{\alpha_l'=0}^{a_l'}\sum_{\beta_l'=0}^{b_l'}\sum_{\gamma_l'=0}^{c_l'} \times\binom{a_l}{\alpha_l}\binom{b_l}{\beta_l}\binom{c_l}{\gamma_l}\binom{a_l'}{\alpha_l'}\binom{b_l'}{\beta_l'}\binom{c_l'}{\gamma_l'} \times\langle\bar{K}',\{\phi'(\alpha_j',\beta_j',\gamma_j')\}|\hat{F}_2|\bar{K}\,\{\phi(\alpha_j,\beta_j,\gamma_j)\}\rangle, \qquad (3.23)$$

where $\langle\bar{K}',\{\phi'(\alpha_j',\beta_j',\gamma_j')\}|\hat{F}_2|\bar{K},\{\phi(\alpha_j,\beta_j,\gamma_j)\}\rangle$ has almost the same formal expression that $\langle U_A^{\bar{K}'}(\{\phi'\}|\hat{F}_2|U_A^{\bar{K}}(\{\phi\})\rangle$ would have if $\bar{K}$ and $\bar{K}'$ were both equal to $K_{\min}$; that is, it is zero when $D_A^{\bar{K}}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})_{\mathbf{k}}$ and $D_A^{\bar{K}'}(\{\phi(\alpha_j',\beta_j',\gamma_j')\})_{\mathbf{k}'}$ have repeated orbitals and/or their sets of orbital labels have less than $A=2$ orbitals in common and, when none of these vanishing conditions prevails, it assumes the value

$$\langle\bar{K}',\{\phi'(\alpha_j',\beta_j',\gamma_j')\}|\hat{F}_2|\bar{K},\{\phi(\alpha_j,\beta_j,\gamma_j)\rangle = \frac{\mathcal{E}NN'A!\,\nu\nu'\pi^{3A/2-2}i^{K-K'}}{2^{K-\bar{K}+K'/2-\bar{K}'/2-1}}\sum_{i,j}\sum_{s=0}^{d_{ij}'} \times[\Gamma((K+K'+3A)/2-3-s)(2s+1)!!]^{-1} \times\int_0^1 dz\, z^{s+1/2}(1-z)^{(K+K')/2-s+3A/2-4}\,v(\rho\sqrt{2z}) \times\{G^{(1)}(i,j;s)[\langle ij|\mathcal{O}(1,2)|ij\rangle+\langle ji|\mathcal{O}(1,2)|ji\rangle] - G^{(2)}(i,j;s)[\langle ij|\mathcal{O}(1,2)|ji\rangle+\langle ji|\mathcal{O}(1,2)|ij\rangle]\}. \qquad (3.24)$$

The value of $\mathcal{E}$ and the range of variation of i and j depend on the number of orbitals that $D_A^{\bar{K}'}(\{\phi'(\alpha_j',\beta_j',\gamma_j')\})_{\mathbf{k}'}$ and $D_A^{\bar{K}}(\{\phi(\alpha_j,\beta_j,\gamma_j)\})_{\mathbf{k}}$ have in common:

(i) $\mathcal{E}=(-)^P$ and $j>i=1,2,\ldots,A$ (3.25)

when the orbitals of one set differ from those of the other set only by a permutation P;

(ii) $\mathcal{E}=(-)^P(-)^{P'}$, $j=A$ and $i=1,2,\ldots,A-1$ (3.26)

when the sets have only $(A-1)$ orbitals in common, brought to the order $1,2,\ldots,A-1$ by the permutations P and P';

(iii) $\mathcal{E}=(-)^P(-)^{P'}$, $j=A$, $i=A-1$ (3.27)

when the sets have only $(A-2)$ orbitals in common, brought to the order $1,2,\ldots,A-2$ by the permutations P and P'.

The other constants of Eq. (3.24) have the same formal expressions of Ref. 17 adapted to this new situation; i.e.,

$$\nu=\left[\prod_{l=1}^{A}(a_l-\alpha_l)!\,(b_l-\beta_l)!\,(c_l-\gamma_l)!\right]^{1/2},$$
$$\nu'=\left[\prod_{l=1}^{A}(a_l'-\alpha_l')!\,(b_l'-\beta_l')!\,(c_l'-\gamma_l')!\right]^{1/2}, \tag{3.28}$$

$$d_{ij}'=(a_i+b_i+c_i+a_j+b_j+c_j+a_i'+b_i'+c_i'+a_j'+b_j'+c_j'$$
$$-\alpha_i-\beta_i-\gamma_i-\alpha_j-\beta_j-\gamma_j-\alpha_i'-\beta_i'-\gamma_i'-\alpha_j'-\beta_j'-\gamma_j)/2, \tag{3.29}$$

$$\langle i'j'|\mathcal{O}(1,2)|ij\rangle=\Lambda^\dagger_{\mu_i',\tau_i'}(1)\,\Lambda^\dagger_{\mu_j',\tau_j'}(2)\,\mathcal{O}(1,2)\,\Lambda_{\mu_i\tau_i}(1)\,\Lambda_{\mu_j\tau_j}(2), \tag{3.30}$$

$$G^{(q)}(i,j;s)=\frac{\xi_{ij}'(-)^s}{2^s s!\,(d_{ij}'-s)!}\sum_{l=0}^{d_{ij}'-s}\binom{d_{ij}'-s}{l}2^{-2l}J_{ij}'^{(q)},\quad q=1,2, \tag{3.31}$$

$$\xi_{ij}'=[(a_i-\alpha_i)!\,(b_i-\beta_i)!\,(c_i-\gamma_i)!\,(a_j-\alpha_j)!\,(b_j-\beta_j)!\,(c_j-\gamma_j)!$$
$$\times(a_i'-\alpha_i')!\,(b_i'-\beta_i')!\,(c_i'-\gamma_i')!$$
$$\times\,(a_j'-\alpha_j')!\,(b_j'-\beta_j')!\,(c_j'-\gamma_j')!]^{-1/2}, \tag{3.32}$$

$$J_{ij}'^{(1)}=\lim_{\{\mathbf{k}_i\},\{\mathbf{k}_i'\}\to 0}\left(\frac{\partial}{\partial k_{x1}}\right)^{a_i-\alpha_i}\left(\frac{\partial}{\partial k_{x1}'}\right)^{a_i'-\alpha_i'}\left(\frac{\partial}{\partial k_{y1}}\right)^{b_i-\beta_i}$$
$$\times\left(\frac{\partial}{\partial k_{y_1}'}\right)^{b_i'-\beta_i'}\left(\frac{\partial}{\partial k_{z1}}\right)^{c_i-\gamma_i}\left(\frac{\partial}{\partial k_{z1}'}\right)^{c_i'-\gamma_i'}\left(\frac{\partial}{\partial k_{x2}}\right)^{a_j-\alpha_j}\left(\frac{\partial}{\partial k_{x2}'}\right)^{a_j'-\alpha_j'}$$
$$\times\left(\frac{\partial}{\partial k_{y2}}\right)^{b_j-\beta_j}\left(\frac{\partial}{\partial k_{y2}'}\right)^{b_j'-\beta_j'}\left(\frac{\partial}{\partial k_{z2}}\right)^{c_j-\gamma_j}\left(\frac{\partial}{\partial k_{z2}'}\right)^{c_j'-\gamma_j'}$$
$$\times(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_1'+\mathbf{k}_2')^{2(l+s)}(\mathbf{k}_1\cdot\mathbf{k}_1'+\mathbf{k}_2\cdot\mathbf{k}_2')^{n-l}, \tag{3.33}$$

$$J_{ij}'^{(2)}=\lim_{\{\mathbf{k}_i\},\{\mathbf{k}_i'\}\to 0}\left(\frac{\partial}{\partial k_{x1}}\right)^{a_i-\alpha_i}\left(\frac{\partial}{\partial k_{x2}'}\right)^{a_i'-\alpha_i'}\left(\frac{\partial}{\partial k_{y1}}\right)^{b_i-\beta_i}$$
$$\times\left(\frac{\partial}{\partial k_{y2}'}\right)^{b_i'-\beta_i'}\left(\frac{\partial}{\partial k_{z1}}\right)^{c_i-\gamma_i}\left(\frac{\partial}{\partial k_{z2}'}\right)^{c_i'-\gamma_i'}\left(\frac{\partial}{\partial k_{x2}}\right)^{a_j-\alpha_j}$$
$$\times\left(\frac{\partial}{\partial k_{x1}'}\right)^{a_j'-\alpha_j'}\left(\frac{\partial}{\partial k_{y2}}\right)^{b_j-\beta_j}\left(\frac{\partial}{\partial k_{y1}'}\right)^{b_j'-\beta_j'}\left(\frac{\partial}{\partial k_{z2}}\right)^{c_j-\gamma_j}$$
$$\times\left(\frac{\partial}{\partial k_{z1}'}\right)^{c_j'-\gamma_j'}(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_1'+\mathbf{k}_2')^{2(l+s)}(\mathbf{k}_1\cdot\mathbf{k}_1'+\mathbf{k}_2\cdot\mathbf{k}_2')^{n-l}. \tag{3.34}$$

It follows from the definition of the B's and from the above equations that the matrix elements (3.8) of the two-body central operators (3.9) vanish when K and K' have opposite parities.

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APPENDIX A: $E_{3(A-1)}$ LAPLACIAN IN TERMS OF ρ_i

The Laplacian in $E_{3(A-1)}$ is defined as

$$\nabla^2=\sum_{i=1}^{A-1}\nabla^2_{\xi_i}, \tag{A1}$$

where the $\boldsymbol{\xi}_i$ are the Jacobi coordinates

$$\boldsymbol{\xi}_i=\left(\frac{i-1}{i}\right)^{1/2}\left(\mathbf{r}_{i+1}-\frac{1}{i}\sum_{j=1}^{i}\mathbf{r}_j\right),\quad i=1,2,\ldots,(A-1), \tag{A2}$$

$$\boldsymbol{\xi}_A=\frac{1}{\sqrt{A}}\sum_{j=1}^{A}\mathbf{r}_j=\sqrt{A}\,\mathbf{R}. \tag{A3}$$

One uses Eqs. (A2) and (A3) to define a matrix C whose entries C_{ij} are given by

$$\boldsymbol{\xi}_i=\sum_{j=1}^{A}C_{ij}\mathbf{r}_j. \tag{A4}$$

Comparing Eq. (A4) with Eqs. (A2) and (A3), it is easy to verify that the matrix C is orthogonal, i.e.,

$$\tilde{C}C=C\tilde{C}=I. \tag{A5}$$

Using Eqs. (A4) and (A5) one obtains

$$\boldsymbol{\rho}_j=\mathbf{r}_j-\mathbf{R}=\sum_{k=1}^{A}(C^{-1})_{jk}\boldsymbol{\xi}_k=\sum_{k=1}^{A}C_{kj}\,\boldsymbol{\xi}_k-\frac{1}{\sqrt{A}}\,\boldsymbol{\xi}_A \tag{A6}$$

from which it follows that

$$\nabla_{\xi_k}=\sum_{j=1}^{A}C_{kj}\nabla_{\rho_j},\quad k=1,2,\ldots,A-1 \tag{A7}$$

and, consequently,

$$\nabla^2=\sum_{k=1}^{A-1}\nabla^2_{\xi_k}=\sum_{k=1}^{A-1}\left(\sum_{i=1}^{A}\sum_{j=1}^{A}C_{ki}C_{kj}\nabla_{\rho_i}\cdot\nabla_{\rho_j}\right)$$
$$=\sum_{i,j=1}^{A}\left(\sum_{k=1}^{A}\tilde{C}_{ik}C_{kj}-C_{Ai}C_{Aj}\right)\nabla_{\rho_i}\cdot\nabla_{\rho_j}$$
$$=\sum_{i,j=1}^{A}\left(\delta_{ij}-\frac{1}{A}\right)\nabla_{\rho_i}\cdot\nabla_{\rho_j}$$
$$=\sum_{i=1}^{A}\nabla^2_{\rho_i}-\frac{1}{A}\left(\sum_{i=1}^{A}\nabla_{\rho_i}\right)\cdot\left(\sum_{j=1}^{A}\nabla_{\rho_j}\right). \tag{A8}$$

APPENDIX B: THE EFFECT OF ONE-BODY OPERATORS ON SLATER DETERMINANTS

Let us denote a Slater determinant with orbitals (2.1) as

$$(\phi_1,\phi_2,\ldots,\phi_A)=\begin{vmatrix}\phi_1(1)&\cdots&\phi_1(A)\\ \vdots&&\vdots\\ \phi_A(1)&\cdots&\phi_A(A)\end{vmatrix} \tag{B1}$$

and define the one-body operator

$$\mathcal{O}=\sum_{i=1}^{A}\mathcal{O}(i), \tag{B2}$$

where $\mathcal{O}(i)$ are operators which act on the relative vector of particle i as well as in its spin–isospin variables.

Using Eqs. (B1) and (B2) one obtains

$$\mathcal{O}(\phi_1,\phi_2,\ldots,\phi_A)$$
$$=\sum_{i_1,i_2,\ldots,i_A}\epsilon_{i_1i_2\cdots i_A}\mathcal{O}[\phi_1(i_1)\phi_2(i_2)\cdots\phi_A(i_A)]. \tag{B3}$$

Now

$$\mathcal{O}[\phi_1(i_1)\phi_2(i_2)\cdots\phi_A(i_A)]$$
$$=[\mathcal{O}(i_1)\phi_1(i_1)]\,\phi_2(i_2)\cdots\phi_A(i_A)$$
$$+\phi_1(i_1)[\mathcal{O}(i_2)\phi_2(i_2)]\,\phi_3(i_3)\cdots\phi_A(i_A)$$
$$\text{-------------------------------}$$
$$\phi_1(i_1)\cdots\phi_{A-1}(i_{A-1})[\mathcal{O}(i_A)\phi_A(i_A)]. \qquad \text{(B4)}$$

Substituting Eq. (B4) into Eq. (B3) one sees that each term of Eq. (B4) leads to a Slater determinant. Therefore, one has

$$\mathcal{O}(\phi_1,\phi_2,\ldots,\phi_A)=(\tilde{\phi}_1,\phi_2,\ldots,\phi_A)+(\phi_1,\tilde{\phi}_2,\phi_3,\ldots,\phi_A)+\cdots+(\phi_1,\phi_2,\ldots,\phi_{A-1},\tilde{\phi}_A), \qquad \text{(B5)}$$

where

$$\tilde{\phi}_j(i)=\mathcal{O}(i)\,\phi_j(i). \qquad \text{(B6)}$$

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Bifurcation from rotationally invariant states[a)]

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Bifurcation in the presence of the rotation group is investigated. The covariant bifurcation equations are derived using the familiar angular momentum operators of quantum mechanics. Variational methods are also discussed. It is shown that the quadratic terms either vanish for odd l or possess a gradient structure for even l. This result is generalized to the case of an arbitrary simply reducible group. Applications to problems in geophysics and elasticity theory are discussed.

1. SYMMETRY BREAKING INSTABILITIES

There are a number of situations in classical mechanics in which the onset of instability of a physical system is accompanied by a spontaneous symmetry-breaking bifurcation. For example, the onset of convection in a spherical mass or the buckling of a perfectly uniform spherical shell leads to a bifurcation which breaks complete rotational symmetry. In such cases one is led to an investigation of the branching of solutions of a nonlinear functional equation $G(\lambda, u) = 0$ in the neighborhood of a known solution (λ_0, u_0). If $G_u(\lambda_0, u_0)$ (G_u denotes the Frechet derivative of G) is a Fredholm operator of index 0, the problem is reduced, via the Lyapounov–Schmidt method, to a finite-dimensional problem

$$F_i(\lambda, z_1, \ldots, z_n) = 0, \quad i = 1, \ldots, n \tag{1.1}$$

where $n = \dim \ker G_u(\lambda_0, u_0)$.

If the original equations $G(\lambda, u)$ are covariant with respect to a representation T_g of a group $\mathcal{G}$—that is, if $T_g\, G(\lambda, u) = G(\lambda, T_g u)$—then the bifurcation equations (1.1) are covariant with respect to a finite-dimensional representation of $\mathcal{G}$. A direct computation of Eqs. (1.1) by numerical methods is often a major obstacle in their analysis, certainly if the original system of equations is very complicated. Using the covariance of the equations, however, the structure of the bifurcation equations can be computed up to unknown constants. In the case of the rotation group Busse,[1] using classical formulas of Gaunt for triple integrals of spherical harmonics, constructed the quadratic terms of (1.1) when $\ker G_u$ transforms according to an even irreducible representation of SO(3). In this paper we give an algorithm for obtaining the full structure of Eqs. (1.1) at all orders based on the Lie algebra of infinitesimal generators of the rotation group. The methods are familiar in the theory of angular momentum coupling in elementary quantum mechanics.

Group theoretic methods allow one to determine the bifurcation equations only up to unknown scalar constants; the dependence of these scalars on the original physical parameters of a particular problem could be determined by a direct computation of the bifurcation equations, say from the Lyapounov–Schmidt method. Rather than proceed in that direction, one can follow an approach similar in spirit to Thom's catastrophe theory[2,3]: The unknown parameters are regarded as free parameters, or control parameters, and one seeks a classification of the types of transitions (i.e., singularities) which may occur. In this way one can obtain a universal classification of the bifurcations which may occur in a physical system which is based on the geometry of the problem and is independent of the particular physical mechanism involved.

In resolving a bifurcation problem one is interested in determining the stability of the bifurcating solutions, and these questions are also discussed in the present paper. Since there is a three-parameter group present, the solutions appear in three- (or sometimes two-) dimensional orbits; hence they will at best be orbitally stable, with two or three neutral modes.

In Sec. 2 we review some of the basic ideas of bifurcation theory, adding some modest improvements to cover the present case. In Sec. 3 we discuss the Lie algebra of angular momentum operators J_+, J_-, and J_3 and show how these may be used to construct Eqs. (1.1) when $\ker G_u$ is irreducible; in Sec. 4 we discuss the modifications which must be made when the kernel is reducible. We also construct the generating function for the number of covariant terms in (1.1) of any given degree. The derivation is closely related to that of the Molien function (Jaric and Birman[4]). Given a finite-dimensional representation Γ of a compact group $\mathcal{G}$, the Molien function counts the number of times the identity representation is contained in the symmetric part of $\Gamma^{\otimes n}$. In the present case we are interested in counting the number of times Γ is contained in the symmetric part of $\Gamma^{\otimes n}$; the generating function in that case is

$$M_1(\Gamma; \mathcal{G}, z) = \int \det[I - z\Gamma(g)]^{-1} \overline{X}(g)\, d\mu(g), \tag{1.2}$$

where $d\mu(g)$ is the normalized invariant measure on $\mathcal{G}$ and X is the character of Γ. We calculate M_1 explicitly for the rotation group O(3).

The extremum principle discovered by Busse[1] is discussed in Sec. 5 and its relationship to the symmetry of the 3-j symbols for SO(3) explained. More generally we show that the result continues to hold whenever Eqs. (1.1) are covariant with respect to an irreducible representation of any simply reducible group. A theorem of Wigner[5,6] on representations of simply reducible groups then implies that the quadratic

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terms in (1.1) vanish for an odd representation or possess a gradient structure for an even representation. This is a striking result, since it implies the bifurcation equations may possess a gradient structure even though the original problem did not arise as a variational problem.

In Sec. 5 we also show that the variational problem associated with the bifurcation equations can, in the case of SO(3), be formulated as

$$\min \tfrac{1}{3}\mathrm{tr}A^3$$

subject to the constraints

$$\tfrac{1}{2}\mathrm{tr}A^2=1, \quad \mathrm{tr}A=0, \quad \mathrm{tr}AB_j=0,$$

where A is a symmetric matrix and the B_j are symmetric matrices which transform according to certain representations of SO(3). For $l=2$ this leads to the Euler–Lagrange equations

$$A^2=\lambda A+\gamma I, \tag{1.3}$$

where A is a 3×3 symmetric traceless matrix and I is the 3×3 identity matrix. This problem is easily resolved, giving an especially simple resolution of the bifurcation problem in the case $l=2$. (The results described in this paragraph were obtained jointly with L. Green) The approach is compared with that discussed by Michel and Radicati[7,8] in their investigations of symmetry breaking in elementary particle physics.

Section 6 contains an analysis of the relationship of the stability properties of the bifurcating solutions to the extremal properties of the solutions of the variational problem. Results of this type have previously been obtained by Sather.[9]

In Sec. 7 we discuss the resolution of the bifurcation equations when $\ker G_u$ transforms according to an irreducible representation D^l of SO(3) for low values of l. Busse's solutions for even l are discussed, and their stability is analyzed.

Finally, in Sec. 7 we discuss situations in classical physics in which questions of bifurcation in the presence of O(3) arise. These are generally problems in geophysics[10] and elasticity theory[11,12,9] which are modeled by nonlinear systems of partial differential equations. We close with a brief discussion of some of the open mathematical problems.

2. LYAPOUNOV-SCHMIDT METHOD

The Lyapounov–Schmidt method, or alternative method, discussed at length by many authors, enables one to reduce an infinite-dimensional problem to a finite-dimensional one. We present here, very briefly, a slight modification of the argument in Ref. 13 which deals with the case in which the equations are covariant with respect to a transformation group.

Suppose the equilibrium states of a physical system are represented by solutions of the nonlinear system of equations

$$G(\lambda,u)=0, \tag{2.1}$$

where $G:\Lambda\times X\to Y$ is a smooth (Frechet differentiable) mapping, Λ is a finite-dimensional vector space, and X and Y are Banach spaces. We assume here that all spaces are Banach spaces over the complex numbers. Let (λ_0,u_0) be a solution pair of (2.1) and let $L_0=G_u(\lambda_0,u_0)$ (G_u denotes the Frechet derivative of G). Let $\mathcal{N}=\ker L_0\subset X$ and $\mathcal{R}=\mathrm{Range}L_0\subset Y$. We assume that G is regular in the sense that for any (λ_0,u_0) $\mathcal{R}$ is always a closed subspace of finite codimension, $\mathcal{N}$ is finite-dimensional, and $\dim\mathcal{N}=\mathrm{codim}\mathcal{R}$. If $\mathcal{N}$ is trivial and $\mathcal{R}=Y$, then by the implicit function theorem there is an analytic curve of solutions $u=u(\lambda)$, defined for sufficiently small $|\lambda-\lambda_0|$, with $u(\lambda_0)=u_0$. From now on, for simplicity, we shall always assume $\lambda_0=0$, $u_0=0$.

If $\mathcal{N}$ is nontrivial, then (0,0) may be a bifurcation point of solutions of (2.1): That is, there may be several distinct solution branches which confluesce at (λ_0,u_0). Let $\dim\mathcal{N}=n$ and choose vectors $\varphi_1^*,\ldots,\varphi_n^*$ in Y^* such that

$$\mathcal{R}=\{f: \langle f,\varphi_j^*\rangle=0, \ j=1,\ldots,n\}.$$

Then the φ_j^* must be null vectors of the adjoint operator L_0^*. Choose vectors $\varphi_1,\ldots,\varphi_n\in Y$ such that $\langle\varphi_i,\varphi_j^*\rangle=\delta_{ij}$; then the linear operator

$$P_2 f=\sum_{j=1}^{n}\langle f,\varphi_j^*\rangle\varphi_j$$

is a projection, and $Q_2=I-P_2$ is a projection of Y onto $\mathcal{R}$. Similarly, let P_1 be the projection onto the kernel $\mathcal{N}(L_0P=0)$. We can write

$$P_1 u=\sum_{j=1}^{n}\langle u,\psi_j^*\rangle\psi_j,$$

where the vectors ψ_j span $\mathcal{N}$. Let $Q_1=I-P_1$.

To reduce (2.1) to a finite-dimensional problem in a neighborhood of (0,0), we decompose the problem as follows:

$$u=P_1u+Q_1u=v+\psi,$$

$$G(\lambda,u)=P_2G(\lambda,u)+Q_2G(\lambda,u)=0.$$

We first solve

$$H(\lambda,v,\psi)=Q_2G(\lambda,v+\psi)=0. \tag{2.2}$$

At the point $\lambda=0$, $v=0$, $\psi=0$, the Frechet derivative of $H(\lambda,v,\psi)$ with respect to ψ is

$$H_\psi(0,0,0)=Q_2G_u(0,0)=Q_2L_0.$$

Now Q_2L_0 is an isomorphism from the subspace Q_1X to Q_2Y. In fact, $Q_2L_0u=0$ implies $L_0u=0$ and therefore that $u\in\mathcal{N}$; but if $u\in Q_1X\cap\mathcal{N}$ then $u=0$. Therefore, L_0 is a bounded one-to-one mapping from Q_1X to Q_2Y. By the closed graph theorem L_0 is invertible, hence an isomorphism. It follows from the implicit function theorem on a Banach space that there is a smooth solution $\psi=\psi(\lambda,v)$ of (2.2). Since X and Y are complex Banach spaces, ψ is analytic in λ and v. The solutions of the full equations (2.1) are obtained now as solutions of the *bifurcation equations*

$$F(\lambda,v)=P_2G(\lambda,v+\psi(\lambda,v))=0. \tag{2.3}$$

Equations (2.3) comprise a system of n equations in n unknowns; by writing $v=z_1\psi_1+\ldots+z_n\psi_n$ we can rewrite (2.3) as

$$F_j \cdot (\lambda, z_1, \ldots, z_n) = \langle G(\lambda, z_1\psi_1 + \ldots + z_n\psi_n + \psi(\lambda, v), \varphi_j^*\rangle = 0.$$

Now suppose the nonlinear mapping G is covariant with respect to a representation T_g of a group $\mathcal{G}$:

$$T_g G(\lambda, u) = G(\lambda, T_g u). \tag{2.4}$$

We have

Theorem 2.1: Let $G(\lambda, u)$ be covariant with respect to a representation T_g of $\mathcal{G}$. Then $\mathcal{N}$ reduces T_g. Assume the projections P_i, Q_i commute with T_g. Then the bifurcation equations themselves are covariant with respect to the finite-dimensional representation $\Gamma = T_g|\mathcal{N}$: that is, $\Gamma F(\lambda, v) = F(\lambda, \Gamma v)$, where F is given by (2.3).

Theorem (2.1) was proved in Ref. 13 where it was shown that commuting projections P_i and Q_i can be constructed if $X \subset Y$. [In that case the resolvent operator $(\lambda - L)^{-1}$ is well defined and the commuting projections can be obtained by the standard residue formula

$$P = \frac{1}{2\pi i}\int_C (\lambda - L)^{-1} d\lambda,$$

where C encloses the isolated eigenvalue of L at the origin.] The assumption $X \subset Y$ is quite natural if G is an elliptic system of partial differential operators; then a natural choice for X and Y is typically $X = C_{m+k+\alpha}$, $Y = C_{m+\alpha}$, where $C_{j+\alpha}$ are the Banach spaces of functions with Hölder continuous derivatives. We note here, nevertheless, that if $\mathcal{G}$ is compact, we can drop the assumption $X \subset Y$ and construct commuting projections as follows.

Lemma 2.2: Let T_g be a representation of the compact group $\mathcal{G}$ on the Banach spaces X and Y. Let L be a bounded mapping from X to Y which intertwines with T_g; $T_g L = L T_g$. Let $\mathcal{N} = \ker L \subset X$ be a closed finite-dimensional subspace and let $R = \text{Range} L \subset Y$ be a closed subspace of finite codimension. Let Q be any projection onto R and $P = I - Q$. Then the projections $\hat{Q}$ and $\hat{P} = I - \hat{Q}$, where

$$\hat{Q} = \int_{\mathcal{G}} T_{g^{-1}} Q T_g \, d\mu(g) \tag{2.5}$$

commute with T_h for all $h \in G$. The same result holds for the projections P_1 and $Q_1 = I - P_1$, where P_1 is any projection onto $\ker L$ in X.

Proof: The fact that $\hat{Q}$ as given in (2.5) commutes with T_h follows from the invariance of the measure $d\mu(g)$. Since $\mathcal{R}$ is invariant under T_g and Q, it is clear that $\mathcal{R}$ is invariant under $\hat{Q}$ as well, and also its range is contained in $\mathcal{R}$. It remains to show that $\hat{Q}$ is a projection, and to that end it is enough to show that $\hat{Q}f = f$ if $f \in R$. We have, whenever $f = Lu$,

$$\hat{Q}f = \hat{Q}Lu = \int T_{g^{-1}} Q T_g L u \, d\mu(g)$$
$$= \int T_{g^{-1}} Q L T_g u \, d\mu(g)$$
$$= \int T_{g^{-1}} L T_g u \, d\mu(g) = Lu = f.$$

The proof for the case that P_1 is a projection onto $\ker L$ in X goes similarly.

We remark that in the case of representations of a noncompact group a reducing subspace need not possess a projection which commutes with the representation. For example, the action of $\mathbb{R}^1$ on $\mathbb{R}^2$ by $\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$ leaves the x axis invariant, but all projections onto the x axis take the form $\left(\begin{smallmatrix}1 & a\\ 0 & 0\end{smallmatrix}\right)$, and none of these commute with the action.

In order to analyze the bifurcation equations (2.3), it is often convenient to reduce them further by scaling them, as follows. A uniformizing parameter ϵ is introduced by setting

$$\lambda = \epsilon^m \sigma, \quad v = \epsilon^n w, \tag{2.6}$$

where $w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots$. The appropriate powers of m and n are determined by a Newton diagram.[13] Now suppose

$$F(\epsilon^m\sigma, \epsilon^n w) = \epsilon^k Q(\sigma, w_0) + O(\epsilon^{k+1}),$$

where $k > \max\{m, n\}$. Dividing by ϵ^k and letting $\epsilon \to 0$, we arrive at the *reduced bifurcation equations*

$$Q(\sigma, w_0) = 0, \tag{2.7}$$

where σ may be chosen conveniently. If the Jacobian $Q_w(\sigma, w_0)$ is invertible at a solution (σ, w_0) of (2.7), then solutions of the full bifurcation equations may be obtained from the implicit function theorem. In the case, however, that Eq. (2.3), hence (2.7), are invariant under a Lie group the solutions of (2.7) may appear in k-parameter sheets; and in that case the Jacobian $Q_w(\sigma, w_0)$ will possess a k-dimensional kernel, spanned by the vectors $L_j w_0$, where the operators L_j are the infinitesimal generators of the Lie group $\mathcal{G}$.

Given a solution (σ, w_0) of (2.7) we examine the full solution curve (2.6). If m is even, the bifurcation is one-sided (that is, solutions appear for $\lambda > 0$ or $\lambda < 0$). When $\sigma > 0$, the bifurcation is supercritical, and it is subcritical when $\sigma < 0$. When m is odd, the branches appear on both sides of criticality (transcritical case).

Stability of the bifurcating solutions: Let a nontrivial one-parameter branch of solutions of (2.1) be given by $(\lambda(\epsilon), u(\epsilon))$ and put $L(\epsilon) = G_u(\lambda(\epsilon), u(\epsilon))$. According to the principle of linearized stability the local stability of the solution $u(\epsilon)$ is determined by the eigenvalues of $L(\epsilon)$. When $\epsilon = 0$, $L(0) = L_0$ has (by assumption) an eigenvalue of multiplicity n at the origin; and, if the trivial solution $u = 0$ is just losing stability as λ crosses zero, all other eigenvalues of L_0 must lie strictly in the left half-plane. The stability of the bifurcating branch is therefore determined by the behavior of the n-fold eigenvalue at the origin as ϵ varies from zero. The following theorem is proved in Ref. 14.

Theorem 2.3: Let $E(\epsilon)$ denote the analytic projection valued operator whose range is the n-dimensional invariant subspace of $L(\epsilon)$ corresponding to the n-fold eigenvalue at the origin. Then the eigenvalues of $L(\epsilon)$ in the vicinity of the origin are precisely those of the n-dimensional operator $B(\epsilon) = L(\epsilon)E(\epsilon)$. Furthermore, if the scaling of the solutions have the form (2.6), then

$$B(\epsilon) = \epsilon^{k-n} Q_w(\sigma, w_0) + O(\epsilon^{k-n+1}).$$

Accordingly, to lowest order in ϵ, the behavior of the multiple eigenvalue 0 under the perturbation along the bifurcating branch is determined by the eigenvalues of the Jacobian of the reduced bifurcation equations. Supercritical solutions are stable if all eigenvalues of $Q_w(\sigma, w_0)$ are negative and subcritical solutions are stable if all eigenvalues of $Q_w(\sigma, w_0)$ are positive.

When a continuous transformation group is present, one or more of the eigenvalues of $Q_w(\sigma, w_0)$ are zero (depending on the dimension of the manifold of solutions); in that case one can at best conclude orbital stability from an analysis of the reduced equations: There will always be a number of neutral modes present.

3. CONSTRUCTION OF THE COVARIANT BIFURCATION EQUATIONS IN THE CASE SO(3)

We denote by Γ the representation $T_g|\mathcal{N}$. Let us expand $F(\lambda, v)$ of (2.3) in a power series in v:

$$F(\lambda, v) = A(\lambda)v + B_2(\lambda, v, v) + B_3(\lambda, v, v, v) + \dots .$$

Then we must have

$$\begin{aligned} &\Gamma A(\lambda)v = A(\lambda)\Gamma v, \\ &\Gamma B_2(\lambda, v, w) = B_2(\lambda, \Gamma v, \Gamma w), \\ &\quad\vdots \end{aligned} \tag{3.1}$$

Therefore, each multilinear operator B is covariant with respect to the representation Γ.

We first make the assumption that Γ is irreducible, that is, that $\Gamma = D^l$, where D^l is one of the irreducible representations of the irreducible representations of SO(3). The contrary case, when $\mathcal{N}$ is reducible, is sometimes called "accidental degeneracy" by physicists (Ref. 5 p. 161); Ruelle[15] suggests the situation is nongeneric. Indeed, that is clearly the case in a problem analyzed in detail by Chow, Hale, and Mallet-Paret.[16] They consider the buckling of a rectangular plate. Since the symmetry group of the rectangle is Abelian, the irreducible representations are all one dimensional; but when the ratio of length to width is $\sqrt{2}$, the principle eigenvalue has multiplicity 2. This situation is clearly nongeneric, for it depends on a specific choice of physical parameters.

The reducible case is discussed in the next section. When Γ is irreducible, the linear term in (3.1) is a scalar multiple of the identity by Schur's lemma. Thus $A(\lambda) = \sigma(\lambda)I$ for some scalar σ. The quadratic term $B(\lambda, v, w)$ must be symmetric in v and w and transform as D^l. The quadratic mapping B may be regarded as a subspace of symmetric second order tensors which transform as D^l under the action of SO(3). The Clebsch—Gordan series

$$D^l \otimes D^l = D^{2l} \oplus D^{2l-1} \oplus \dots \oplus D^0 \tag{3.2}$$

tells us that the tensor product space $\mathcal{N} \otimes \mathcal{N}$ decomposes into a direct sum of subspaces, precisely one of which transforms according to D^l, as follows:

$$\mathcal{N} \otimes \mathcal{N} = V^{2l} \oplus \dots \oplus V^l \oplus \dots \oplus \dots \oplus V^0.$$

In this decomposition V^{2l} consists of symmetric tensors, V^{2l-1} antisymmetric tensors, and so forth. Accordingly V^l consists of symmetric tensors *iff* l is even. Therefore, *for odd* l *the quadratic term vanishes*, and we must go to cubic terms to get the reduced bifurcation equations. (We shall show below that a similar result holds more generally when $\mathcal{G}$ is a simply reducible group.)

Since we are interested solely in symmetric tensors over $\mathcal{N}$, we can work with polynomials (due to the natural isomorphism between the ring of polynomials and symmetric tensors over $\mathcal{N}$). We therefore identify $\mathcal{N}$ with the vector space of linear polynomials in the variables $z_{-l}, \dots, z_l$ [since $\dim D^l = (2l+1)$] and denote by $K[z_{-l}, \dots, z_l]$ the ring of polynomials in the independent variables $z_{-l}, \dots, z_l$. K is then isomorphic to the algebra of symmetric tensors over $\mathcal{N}$.

Let the bifurcation equations be

$$F_m(z_{-l}, \dots, z_l) = 0, \quad m = -l, \dots, l.$$

The linear terms of F_m are of the form

$$F_m(z_{-l}, \dots, z_l) = a z_m$$

since, as we have said, the linear term must be a scalar multiple of the identity. The quadratic terms are given by

$$F_m = \sum_{m_1+m_2=m} C(l, m_1; m_2; l, m_2; l, m) z_{m_1} z_{m_2}, \tag{3.3}$$

where $C(l, m_1; l, m_2; l, m)$ are the *Clebsch—Gordan* coefficients for SO(3), or

$$F_m = (-1)^m \sum_{m_1+m_2=m} \begin{pmatrix} l & l & l \\ m_1 & m_2 & -m \end{pmatrix} z_{m_1} z_{m_2}, \tag{3.4}$$

where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

are the Wigner 3-j coefficients for SO(3).

In the general case the terms F_m can be constructed by the following algorithm. Let the infinitesimal generators of SO(3) be J_1, J_2, J_3; these satisfy the commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k,$$

where ϵ_{ijk} is the completely antisymmetric tensor.

Putting $J^{\pm} = \pm J_2 + iJ_1$ and $J^3 = -iJ_3$, we obtain instead the commutation relations for sl(2)

$$[J^+, J^-] = 2J^3, \quad [J^3, J^{\pm}] = \pm J^{\pm}. \tag{3.5}$$

By well-known algorithms (see Ref. 17, p. 234), using the commutation relations (3.5), we can construct a basis f_m for the complexification of the vector space $\mathcal{N}$ such that

$$J_3 f_m = m f_m,$$

$$J_{\pm} f_m = \beta_{\pm m} f_{m\pm 1},$$

where

$$-l \le m \le l \quad \text{and} \quad \beta_m = [(l-m)(l+m+1)]^{1/2}.$$

In addition, the f_m can be normalized so that the reality condition

$$\bar{f}_m = (-1)^m f_{-m} \tag{3.6}$$

is satisfied.

Since I have not found (3.6) in the standard references I will give a proof here. First note that the operators J_1, J_2, J_3 are real operators, and so $\overline{J^3}=-J^3$, $\overline{J^+}=-J^-$, $\overline{J^-}=-J^+$. It follows that $\overline{J^3f_m}=\overline{mf_m}=-J^3\overline{f_m}$ and therefore that $J_3\overline{f_m}=-m\overline{f_m}$. The vector $\overline{f_m}$ has weight $-m$. Since $\mathcal{N}$ is irreducible there is only one vector with weight $-m$, and that is f_{-m}. So $\overline{f_m}=c_mf_{-m}$. On the other hand $J^+f_m=\beta_mf_{m+1}$ so $\overline{J^+f_m}=\beta_m\overline{f_{m+1}}=\beta_mc_{m+1}f_{-(m+1)}$ $=-J^-\overline{f_m}=-J^-c_mf_{-m}=-c_mJ^-f_{-m}=-c_m\beta_mf_{-m-1}$. Consequently, $c_{m+1}=-c_m$ and we can take $c_m=(-1)^mc$. For $m=0$ we have $\overline{f_0}=cf_0$. Choosing $c=1$, we obtain that f_0 is real and $\overline{f_m}=(-1)^mf_{-m}$.

The reality condition (3.6) is important when we wish to restrict ourselves to real solutions of the bifurcation equations (1.1).

We now require the variables z_m to transform as the f_m under J_3 and $J_\pm$. We extend J_3 and $J_\pm$ to be derivations over K:

$$J(\alpha f+\beta g)=\alpha Jf+\beta Jg,$$

$$J(fg)=fJg+(Jf)g,$$

where $f,g\in K$ and α and β are scalars. It is natural to extend the J's in this way since they are Lie derivatives.

If the functions $F_m(z_{-l},\ldots,z_l)$ are to transform as D^l they also must transform as the z_m:

$$J_3F_m=mF_m,\quad J_\pm F_m=\beta_{\pm m}F_{m\pm1}. \tag{3.7}$$

For example, the quadratic polynomials F_m are obtained as follows. The action of J_3 on z_jz_k is

$$J_3(z_jz_k)=(J_3z_j)z_k+z_j(J_3z_k)=(j+k)z_jz_k$$

so $J_3z_jz_k=mz_jz_k$ if $j+k=m$. Therefore,

$$F_m(z_{-l},\ldots,z_l)=\sum_{m_1+m_2=m}A_{m_1m_2m}z_{m_1}z_{m_2}.$$

In particular, when l is even,

$$F_l=a_0z_lz_0+a_1z_{l-1}z_1+\cdots+a_{l/2}(z_{l/2})^2.$$

Furthermore, $J_+F_l=\beta_lF_l=0$, and this condition gives us a set of linear equations for the coefficients $a_0\cdots a_{l/2}$. In the case $l=2$ we have

$$F_2=az_2z_0+bz_1^2,$$

$$J_+F_2=a\beta_0z_2z_1+2b\beta_1z_1z_2$$

$$=(a\beta_0+2b\beta_1)z_1z_2=0,$$

$$a\beta_0+2b\beta_1=0.$$

The last equation determines the coefficients a and b, hence F_2, up to a scalar multiple. Once F_l is known we get F_{l-1} from

$$J_-F_l=\beta_{-l}F_{l-1}$$

and so forth. In this way we can construct all the F_m's.

This procedure extends immediately to higher order terms. For example, to get third order terms we write

$$F_l=\sum_{i+j+k=l}a_{ijk}z_iz_jz_k$$

and apply

$$J_+F_l=0$$

to get a linear system of equations for the a_{ijk}. For $l=1$ there is only one solution but for $l=3$ there are two independent solutions. In fact, the condition $J_+F_l=0$ in that case leads to five equations in seven unknowns (see Sec. 7).

4. THE CASE Γ REDUCIBLE

We begin by deriving a generating function which gives the number of covariant n-linear symmetric mappings for arbitrary n. We first derive a general formula for arbitrary compact groups, and then apply the formula in the specific case of O(3). We denote the irreducible representations of $\mathcal{G}$ by Γ_ν and suppose $\Gamma=\Sigma a_\nu\Gamma^\nu$, where a_ν is the multiplicity of Γ^ν in Γ. The characters of Γ and Γ^ν are denoted by χ and χ^ν respectively.

Theorem 4.1: Let Γ be a representation on the vector space $\mathcal{N}$ of the compact group $\mathcal{G}$ and let $c_n=c_n(\Gamma,\mathcal{G})$ denote the number of completely symmetric n-linear operators B on $\mathcal{N}\otimes\mathcal{N}$ to $\mathcal{N}$ which are covariant with respect to Γ. Then a generating function for the coefficients c_n is

$$\sum_{n=0}^{\infty}c_n(\Gamma,\mathcal{G})z^n=M_1(\mathcal{G},\Gamma,z)=\int_{\mathcal{G}}\det[I-z\Gamma(g)]^{-1}\overline{\chi}(g)\,d\mu(g). \tag{4.1}$$

In the above expression $\mu(g)$ denotes the normalized invariant measure on $\mathcal{G}$; we set $c_0(\Gamma,\mathcal{G})=1$ by convention.

Proof: Let $\mathcal{N}^*$ be the dual of $\mathcal{N}$; let the n-linear map B be covariant with respect to Γ; and put

$$F(u_1,\ldots,u_n;u_{n+1})=\langle B(u_1,\ldots,u_n),u_{n+1}\rangle, \tag{4.2}$$

where $u_{n+1}\in\mathcal{N}^*$ and $\langle\ ,\ \rangle$ denotes the bilinear pairing between $\mathcal{N}$ and $\mathcal{N}^*$. Let $\tilde{\Gamma}(g)=\Gamma^+(g^{-1})$ be the contragradient representation. (Here Γ^+ denotes the adjoint of Γ relative to the bilinear pairing $\langle\ ,\ \rangle$). F is a tensor in $\mathcal{N}^{\otimes n}\otimes\mathcal{N}^*$ which is invariant under the representation $\Gamma^{\otimes n}\otimes\tilde{\Gamma}$; in fact,

$$\Gamma^{\otimes n}\otimes\tilde{\Gamma}F(u_1,\ldots,u_n;u_{n+1})$$
$$=F(\Gamma u_1,\ldots,\Gamma u_n;\tilde{\Gamma}u_{n+1})$$
$$=\langle B(\Gamma u_1,\ldots,\Gamma u_n),\tilde{\Gamma}u_{n+1}\rangle$$
$$=\langle\Gamma B(u_1,\ldots,u_n),\tilde{\Gamma}u_{n+1}\rangle$$
$$=\langle B(u_1,\ldots,u_n),u_{n+1}\rangle=F(u_1,\ldots,u_n;u_{n+1}).$$

The correspondence (4.2) between covariant n-linear maps and $(n+1)$-linear invariants is one—one. On the other hand, the number of invariants is equal to the number of times the identity representation is contained in the tensor product representation $\Gamma^{\otimes n}\otimes\tilde{\Gamma}$. This number is given by the expression

$$\int_{\mathcal{G}}\chi^n(g)\overline{\chi}(g)\,d\mu(g).$$

Now, however, we must modify the argument to take into account the fact that B is completely symmetric. We do this by restricting the representation $\Gamma^{\otimes n}$ to the symmetric part of $\mathcal{N}^{\otimes n}$; the restriction of $\Gamma^{\otimes n}$ to this subspace is denoted by $(\Gamma^{\otimes n})s$. The character $\chi_{(n)}$ of $(\Gamma^{\otimes n})s$ is given by the generating function

$$\det[I - z\Gamma(g)]^{-1} = \sum_{n=0}^{\infty} \chi_n(g) z^n, \tag{4.3}$$

where $\chi_{(0)}(g) \equiv 1$. The result (4.1) now follows immediately.

A derivation of (4.3) may be found in Littlewood[18] in the chapter on Schur functions: but for completeness I will present a simpler derivation in the Appendix.

The expression (4.2) for the covariant mappings is very closely related to the Molien function (see Jarić and Birman[4]) which counts the number of completely symmetric invariant tensors of each order. The Molien function has been calculated by Jarić and Birman for various space groups. Let us calculate the function (4.1) in the case $\mathcal{G} = \mathrm{O}(3)$. First note that the determinant of the direct sum $A \oplus B$ of two matrices is $\det A \oplus B = (\det A)(\det B)$; for the determinant of an operator is the product of its eigenvalues, and the eigenvalues of $A \oplus B$ are the union of those of A and of B. Therefore

$$\det(I - z\sum a_\nu \Gamma_\nu)^{-1} = \prod_\nu [\det(I - z\Gamma_\nu)]^{-a_\nu},$$

where I on the left is the $N\times N$ identity matrix ($N = \dim\Gamma$) and the I's on the right are the $\mathcal{N}_\nu \times \mathcal{N}_\nu$ identity matrices ($\mathcal{N}_\nu = \dim\Gamma_\nu$).

It remains to calculate $\det(I - z\Gamma_\nu)$ for the irreducible representations O(3). We first carry out the calculation for SO(3) and then indicate the modifications which must be made to treat O(3). Let g be a rotation through an angle θ. The eigenvalues of $D^l(g)$ are then $\exp(\pm im\theta$, $m = -l, \ldots, l$, and therefore

$$\det(I - zD^l) = \prod_{m=-l}^{l} (1 - z\exp(im\theta))(1 - z\exp(-im\theta))$$

$$= \left(\prod_{m=0}^{l} (1 - 2z\cos m\theta + z^2)\right).$$

The invariant integral for SO(3) is

$$\frac{1}{\pi}\int_0^\pi (1 - \cos\theta)d\theta$$

and so our expression for (4.1) is

$$M(\mathrm{SO}(3);\Gamma;z) = \frac{1}{\pi}\int_0^\pi \prod_l \prod_{m=0}^{l} (1 - 2z\cos m\theta + z^2)^{-a_l} \times \sum_l a_l \chi_l(\theta)(1 - \cos\theta)d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod_l \prod_{m=0}^{l} (1 - 2z\cos m\theta + z^2)^{-a_l} \times \sum_l a_l \chi_l(\theta)(1 - \cos\theta)d\theta, \tag{4.4}$$

which can be evaluated by residues.

As to O(3) there are two types of representations, positive and negative, which are closely related to those of SO(3). (See Miller,[17] p. 249). When g is a pure rotation, $D_+^l(g) = D_-^l(g) = D^l(g)$; but when g is a rotation reflection, $D_+^l(g) = -D_-^l(g) = D^l(g)$. The negative representations thus contain the inversion $v \to -v$. Since the spherical harmonics satisfy $Y_m^l(\theta, l) = (-1)^l Y_m^l(\pi - \theta, \pi + \varphi)$ the subspaces V^l transform according to positive representations for even l, and negative for odd l. In order to correct (4.4) for the case O(3), we cut (4.4) in half and add another term corresponding to the integral over the rotation-reflections of the group. For this portion of the integral the eigenvalues of the representations are multiplied by a factor of $(-1)^l$. Therefore, the correction term is

$$\frac{1}{2\pi}\int_0^\pi \prod_l \prod_{m=0} (1 - 2z(-1)^l\cos m\theta + z^2)^{-a_l} \times \sum_l a_l(-1)^l \chi_l(\theta)(1 - \cos\theta)d\theta.$$

Now let us turn to algorithms for constructing the covariant bifurcation equations in the reducible case. Such an algorithm was given for the case of a finite group in Ref. 13. Here we present a method using again the Lie algebra of infinitesimal operators. We look first at two simple examples, from which the general algorithm will be clear.

Suppose first that the kernel $\mathcal{N}$ transforms according to the representation $D^1 \oplus D^2$, and represent the vector space $\mathcal{N}$ as linear polynomials in the variables $x_0, x_{\pm1}$ and $y_0, y_{\pm1}, y_{\pm2}$. We then seek polynomials $F_0, F_{\pm1}$ and $G_0, G_{\pm1}, G_{\pm2}$ in these variables which transform as D^1 and D^2 respectively. Let $J_\pm$ and J_3 be the operators discussed in Sec. 3. We have

$$J_3 x_j = jx_j, \quad J_3 y_j = jy_j, \quad J_3 F_j = jF_j, \quad J_3 G_j = jG_j,$$

$$J_+ x_j = \beta_{l,j} x_{j+1}, \quad J_- x_j = \beta_{l,-j} x_{j-1},$$

$$J_\pm y_k = \beta_{2,\pm j} y_{j\pm1},$$

$$J_\pm F_j = \beta_{1,k} F_{j\pm1}, \quad J_\pm G_j = \beta_{2,j} G_{j\pm1}$$

where $\beta_{l,j} = [(l-j)(l+j+1)]^{1/2}$.

Since F_1 must have weight 1, we write it as a sum of all possible terms of weight 1. For the quadratic case we have

$$F_1 = ay_1x_0 + bx_0y_1 + cy_2x_{-1},$$

and we require $J_+F_1 = 0$. We have omitted such terms as x_1x_0 and y_1y_0 because D^1 is not contained in $(D^1 \otimes D^1)s$ or $(D^2 \otimes D^2)s$. (In fact, $D^1 \otimes D^1 = D^2 \oplus D^1 \oplus D^0$, where the first and third invariant subspaces are symmetric tensors, and the subspace which transforms according to D^1 is antisymmetric. A similar situation is true in the case $D^2 \otimes D^2$.) Terms such as x_ix_j show up in the representation $D^1 \otimes D^1$; terms y_iy_j come from $D^2 \otimes D^2$; and terms x_iy_j come from $D^1 \otimes D^2$.

The condition $J_+F_1 = 0$ leads to the equations

$$b\beta_{2,1} + c\beta_{1,-1} = 0, \quad a\beta_{2,0} + b\beta_{1,0} = 0,$$

of which there is one solution. Similarly, for G_2 we take

$$G_2 = ax_1^2 + by_2y_0 + cy_1^2 + dx_1y_1 + ex_0y_2$$

and apply the condition $J_+G_2 = 0$. This leads to the equations $d\beta_{2,1} + e\beta_{1,0} = 0$, $b\beta_{2,0} + 2c\beta_{2,1} = 0$, and no restriction on a. We get three linearly independent solutions in all, so there are four covariant polynomials of degree 2: one of weight 7 and three of weight 2. There are therefore four parameters in the reduced bifurcation equations.

For the second example consider the case that $\mathcal{N}$

transforms as $2D^1$, or $D^1 \oplus D^1$. In this case choose variables $x_0, x_{\pm 1}$ and $y_0, y_{\pm 1}$, and look for polynomials F_1 of weight one such that $J_+F_1 = 0$. The reader will easily determine that there is only one such polynomial, namely $(x_1 y_0 - x_0 y_1)$. This may be repeated *twice* in the bifurcation equations, so there are two parameters. The reduced bifurcation equations are therefore

$$x_1 = A(x_1 y_0 - x_0 y_1), \quad x_0 = A\sqrt{2}(x_1 y_{-1} - x_{-1} y_1),$$
$$x_{-1} = A(x_0 y_1 - x_{-1} y_0), \quad y_1 = B(x_1 y_0 - x_0 y_1),$$
$$y_0 = B\sqrt{2}(x_1 y_{-1} - x_{-1} y_1), \quad y_{-1} = B(x_0 y_{-1} - x_{-1} y_0).$$

The general algorithm now follows. When $\Gamma = \sum a_l D^l$ put $N = \sum a_l$ and choose N different sets of variables $x, y, \cdots$ and N different sets of polynomials $F, G, \cdots$. Each set of variables and polynomials is to transform irreducibly under the Lie algebra so(3) – with a one–one correspondance among variables, polynomials, and the irreducible representations D^l occurring in Γ. Each chain of polynomials of a given weight can occur in any part of the bifurcation equations of the same weight. Thus, to a_ν occurrences of D^ν and b_ν covariant polynomial chains of weight ν there correspond $a_\nu b_\nu$ independent parameter in the bifurcation equations—that is, $a_\nu b_\nu$ independent occurrences of the polynomials of weight ν.

5. GRADIENT STRUCTURE OF THE BIFURCATION EQUATIONS; SIMPLY REDUCIBLE GROUPS

Suppose that the kernel $\mathcal{N}$ is irreducible and that the reduced bifurcation equations take the simple form

$$\sigma w + B(w, w) = 0. \tag{5.1}$$

As we have already seen, $\mathcal{N}$ must transform according to an even representation of O(3) $[D^l(g)$, where l is even$]$, for otherwise the quadratic term is antisymmetric. In that case Eq. (5.1) possess a gradient structure, as Busse[1] has observed. This gradient structure is a consequence of a symmetry property of the 3-j symbols which holds in the more general context of a "simply reducible group."

Recall that the quadratic terms of the bifurcation equations are given (for even l) by

$$F_m(z_{-l}, \ldots, z_l) = \sum (-1)^m \begin{pmatrix} l & l & l \\ m_1 & m_2 & -m \end{pmatrix} z_{m_1} z_{m_2}$$

(The 3-j symbols vanish whenever $|m| > l$ or $m_1 + m_2 + m_3 \neq 0$, hence we may drop the limits of summation.) Consider the homogeneous polynomial of degree 3

$$p(z_{-l}, \ldots z_l) = \tfrac{1}{3} \sum_{-l}^{l} F_m \overline{z_m}$$

restricted to the real subspace of $\mathcal{N}$ for which $\overline{z_m} = (-1)^m z_{-m}$. There we have

$$p(z_{-l}, \ldots z_l) = \tfrac{1}{3} \sum_{-l}^{l} (-1)^m F_m z_{-m}$$
$$= \tfrac{1}{3} \sum_{m_1, m_2, m_3 = -l}^{l} \begin{pmatrix} l & l & l \\ m_1 & m_2 & m_3 \end{pmatrix} z_{m_1} z_{m_2} z_{m_3}.$$

For l even the 3-j symbols are completely symmetric in the integers m_1, m_2, m_3, (Ref. 5, p. 159), and therefore

$$\frac{\partial p}{\partial z_m} = F_m(z_{-l}, \ldots, z_l).$$

In as much as (5.1) can be written in the component form as

$$\sigma z_m + F_m(z_{-l}, \ldots, z_l) = 0, \tag{5.1′}$$

we see that these equations possess a gradient structure and in fact are the Euler–Lagrange equations for the variational problem

$$\min_{|z|=1} \tfrac{1}{3} p,$$

where

$$|z|^2 = \sum_{m=-l}^{l} z_m \overline{z_m} = \sum_{-l}^{l} (-1)^m z_m z_{-m}.$$

The function p is the third order invariant for the representation $D^l(g)$ of O(3); that is, $p(D^l(g)z) = p(z)$ for all $g \in$ O(3). The norm $|z|^2$ is the second order invariant. In vector form these invariants take the form $\langle u, u\rangle$ and $\langle B(u,u), u\rangle$, where $\langle\ ,\ \rangle$ denotes the scalar product on the vector space $\mathcal{N}$. The variational problem is accordingly

$$\min_{\langle u,u\rangle=1} \tfrac{1}{3}\langle B(u,u), u\rangle. \tag{5.2}$$

In general, let $\mathcal{N}$ be a complex inner product space, T_g a unitary representation, and B a covariant symmetric bilinear mapping from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$. The trilinear form $F(u,v,w) = \langle B(u,v), w\rangle$ is always an invariant of $T^{\otimes 3}$, as we saw in Sec. 4. Under what conditions, however, is $B(u,u)$ also the gradient of the function $\mathcal{J}(u) = \frac{1}{3}F(u,u,u)$? The answer is given in the following lemma.

Lemma 5.1: Let B, F and $\mathcal{N}$ be as above. Then B is the gradient of the functional $\mathcal{J}$ iff the trilinear form $\langle B(u, v), w\rangle$ is completely symmetric.

Proof: The operator $B(u)$ is the gradient of the functional $\mathcal{J}(u)$ if $\mathcal{J}(u + \epsilon h) = \mathcal{J}(u) + \epsilon\langle B(u), h\rangle + O(\epsilon^2)$. First suppose the trilinear form $\langle B(u,v), w\rangle$ is completely symmetric. Then

$$\mathcal{J}(x + \epsilon h)$$
$$= \mathcal{J}(x) + (\epsilon/3)[\langle B(h,x), x\rangle + \langle B(x,h), x\rangle + \langle B(x,x), h\rangle]$$
$$+ O(\epsilon^2)$$
$$= \mathcal{J}(x) + \epsilon\langle B(x,x), h\rangle + O(\epsilon^2).$$

By definition, then, the gradient of $\mathcal{J}$ is the bilinear operator B.

The converse is a little harder to prove. By the symmetry of B,

$$\mathcal{J}(x + \epsilon h) = \mathcal{J}(x) + \epsilon[\langle B(x,x), h\rangle + 2\langle B(h,x), x\rangle] + O(\epsilon^2),$$

whereas, if B is the gradient of $\mathcal{J}$,

$$\mathcal{J}(x + \epsilon h) = \mathcal{J}(x) + \epsilon\langle B(x,x), h\rangle + O(\epsilon^2).$$

Comparing terms of order ϵ, we have

$$\langle B(x,x), h\rangle = \langle B(x,h), x\rangle \tag{5.3}$$

for all vectors x and h. Replacing x by $x + y$ in (5.3) and using the symmetry of B, we obtain the identity

$$\langle B(x,y), h\rangle = \tfrac{1}{2}[\langle B(x,h), y\rangle + \langle B(y,h), x\rangle]. \tag{5.4}$$

Interchanging x and h in (5.4), we get

$$\langle B(h,y),x\rangle = \tfrac{1}{2}[\langle B(h,x),y\rangle + \langle B(y,x),h\rangle]$$
$$= \tfrac{1}{2}[\langle B(h,x),y\rangle + \tfrac{1}{2}[\langle B(x,h),y\rangle + \langle B(y,h),x\rangle\}]$$
$$= \tfrac{3}{4}\langle B(x,h),y\rangle + \tfrac{1}{4}\langle B(y,h),x\rangle;$$

hence $\langle B(y,h),x\rangle = \langle B(h,y),x\rangle = \langle B(x,h),y\rangle$. Consequently, the trilinear form F is invariant under the permutations (12) and (13) and so is completely symmetric.

Summarizing the above discussion, we have

Theorem 5.2: When the kernel $\mathcal{N}$ transforms according to the irreducible representation $D^l(g)$ of SO(3) the quadratic term in the bifurcation equations vanishes altogether for l odd and is the gradient of the third order invariant if l is even.

The above theorem continues to hold, with appropriate modifications, in the more general context of simply reducible groups. Simply reducible groups (S.R. groups) were introduced by Wigner in 1940. (References for the following remarks may be found in Wigner[6] and Hammermesh,[5] pp. 151–59). A group is simply reducible if

(a) Every element is equivalent (conjugate) to its inverse (i.e., for every p there is an h such that $p = hp^{-1}h^{-1}$).

(b) The tensor product of any two irreducible representations contains no irreducible representation more than once.

Many of the groups occurring in applications are S.R. groups: the symmetric groups S_3 and S_4, the quaternion group, the three-dimensional rotation group, the two-dimensional unimodular group, and most of the crystal point groups. An immediate consequence of property a is that all the group characters are real [since $\chi(g^{-1}) = \overline{\chi(g)}$ and the character is constant on conjugacy classes] and so every representation is equivalent to its complex conjugate representation.

The irreducible representations of a compact group can be classified into three groups: Those which possess a real matrix representation; those which do not possess a real representation but which are nevertheless unitarily equivalent to their complex conjugate representation; and those which are not equivalent to their complex conjugate representations. Representations of the first kind are called *integer representations*; those of the second are called *half-integer* representations.

Lemma 5.3: The tensor product of two integer representations or of two half-integer representations of an S.R. group contains only integer representations, while the tensor product of an integer and a half-integer representation contains only half-integer representations.

The proof of this lemma is given in Wigner's article.[6]

The tensor product of a representation with itself can be decomposed into a symmetric and an antisymmetric part. For a representation T denote the symmetric part of $T\otimes T$ by $(T\otimes T)$s and the antisymmetric part by $(T\otimes T)$a. If T is an integer representation, the irreducible parts of $(T\otimes T)$s are called *even* representations and and those in $(T\otimes T)$a are called *odd* representations; on the other hand, this nomenclature is reversed if T is a half-integer representation. Wigner has proved that no representation can be simultaneously odd and even, but there are integer representations which are neither even nor odd.

Let the irreducible representations be denoted by D^j and introduce the convention that $(-1)^j = 1(-1)$ if j is an even (odd) representation and $(-1)^{2j} = 1(-1)$ if j is an integer (half-integer) representation.

Theorem 5.4: (Wigner[6] p. 92) For an S.R. group it is possible to normalize the 3-j symbols in such a way that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_2 & j_1 & j_3 \\ \kappa_2 & \kappa_1 & \kappa_3 \end{pmatrix}.$$

Hence the 3 $-j$ symbols remain unchanged under an even permutation of the columns but are multiplied by $(-1)^{j_1}(-1)^{j_2}(-1)^{j_3}$ for an odd permutation.

The following is an immediate consequence of Wigner's theorem.

Theorem 5.5: Let $\mathcal{G}$ be a simply reducible group and let D^j be an irreducible (unitary) integer representation of $\mathcal{G}$ acting on the vector space $\mathcal{N}$. Let the bilinear mapping B be covariant with respect to D^j. Then the third order trilinear invariant $\langle B(u,v),w\rangle$ is completely symmetric if D^j is an even representation and completely antisymmetric if D^j is an odd representation. Consequently, the quadratic terms of the bifurcation equations vanish for an odd representation and possess a gradient structure for even representations.

More generally,

Theorem 5.6: Let Γ be a unitary representation on a Hilbert space $\mathcal{H}$ such that $\Gamma^{\otimes(n+1)}$ contains the identity representation precisely once. Then there is a convariant n-linear map $B:\mathcal{H}\to\mathcal{H}$ which is either completely antisymmetric or is completely symmetric and the gradient of a completely symmetric invariant of order $(n+1)$.

Proof: Since $\Gamma^{(n+1)}$ contains the identity representation once there exists a unique invariant $F(x_1, \ldots, x_{n+1})$. Define a representation of S_{n+1} by $T_\sigma F(x_1, \ldots, x_{n+1}) = F(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n+1)})$ for $\sigma\in S_{n+1}$. Since the subspace of invariants is one-dimensional, $T_\sigma F = \chi_{(\sigma)}F$ where χ is a character of S_{n+1}. Since the only one dimensional representations of the symmetric group are the identity and alternating actions, F is either completely symmetric or completely antisymmetric. The associated covariant operator B is therefore the same. Remark: Given the $(n+1)$ linear form F the mapping B is obtained as follows: Fix $x_1, \ldots, x_n$ and consider the linear functional $u \to F(x_1, \ldots, x_n, u)$. This may be represented as $u \to (B(x_1, \ldots, x_n), u)$ where $B(x_1, \ldots, x_n)\in\mathcal{H}$. The linearity and transformation properties of B are readily derived.) Finally, if F is completely symmetric, then its gradient is easily seen to be the mapping $x \to (n+1)B(x, \ldots, x)$ by the argument of the first part of Lemma 5.1. Theorem 5.5 may be obtained directly from Theorem 5.6 without recourse to Wigner's theorem. For if Γ is an irreducible integer representation of a simply reducible group, then it is equivalent to its contragradient. If Γ is contained in $\Gamma \otimes \Gamma$ precisely once, then the identity representation is contained precisely once in $\Gamma^{\otimes 3}$ ($R \otimes S$ contains the identity representation precisely once iff R and S are unitarily equivalent irreducible representations; see Ref. 13, Theorem 8.1). If Γ is an even representation, it is contained in $(\Gamma \otimes \Gamma)s$, hence the third order invariant must be symmetric, while if Γ is odd it is contained in $(\Gamma \otimes \Gamma)a$ and the third order invariant is antisymmetric.

In the case of the rotation group Professor L. Green (School of Mathematics) and I have succeeded in casting the variational problem in slightly different way. For even l we have the Clebsch–Gordan series

$$D^{l/2} \otimes D^{l/2} = D^l \oplus D^{l-1} \oplus \cdots \oplus D^0 \tag{5.5}$$

and the associated representation

$$U_g A = D^{l/2}(g) A D^{l/2}(g^{-1}) \tag{5.6}$$

on $(l+1)\times(l+1)$ matrices A. This representation is unitary relative to the inner product

$$\langle A, B\rangle = \tfrac{1}{2}\mathrm{tr} AB^* \tag{5.7}$$

(B^* = Hermitian conjugate of B). The third order invariant (there is only one, since $D^l \otimes D^l \otimes D^l$ contains D^0 only once) is

$$p(A) = \tfrac{1}{3}\,\mathrm{tr} A^2 A^*.$$

The highest weight space, the one that transforms like D^l in (5.5), consists of symmetric tensors, so we may restrict ourselves to Hermitian symmetric matrices and rephrase our variational problem as

$$\min \tfrac{1}{3}\,\mathrm{tr} A^3$$

subject to

$$\tfrac{1}{2}\,\mathrm{tr} A^2 = 1 \ \text{ and } \ \mathrm{tr} AB_j = 0,$$

where the B_j are symmetric matrices which lie in the lower weight invariant subspaces. In particular $\mathrm{tr} AI = \mathrm{tr} A = 0$. For $l=2$, (5.5) reads $D^1 \otimes D^1 = D^2 \oplus D^1 \oplus D^0$; but the tensors transforming according to D^1 are antisymmetric, so we have only the constraint $\mathrm{tr} A = 0$, $\mathrm{tr} A^2 = 2$. The Euler–Langrange equations are therefore

$$A^2 = \lambda A + \gamma I, \tag{5.8}$$

where A and I are 3×3 matrices. (The gradient of the functional $\frac{1}{3}\mathrm{tr} A^3$ is the mapping $A \to A^2$). Equations (5.8) can be completely solved as follows.

Taking the trace of (5.8), we get $\gamma = 2/3$. In the case $l=2$, A is a 3×3 matrix and we can choose a rotation g as that $D^1(g) A D^1(g^{-1})$ is diagonal, since $D^1(g)$ ranges over over all orthogonal matrices as g ranges over O(3). So, assuming A is diagonal, we can write (5.8) as

$$\mu_i^2 = \lambda \mu_i + \tfrac{2}{3}, \quad i = 1, 2, 3, \tag{5.9}$$

where μ_1, μ_2, μ_3 are the eigenvalues of A. The constraints are

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = 2, \tag{5.9a}$$

$$\mu_1 + \mu_2 + \mu_3 = 0, \tag{5.9b}$$

There are two sets of solutions to (5.9), (5.10); viz.,

$$\lambda = -1/\sqrt{3}, \quad A = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

and

$$\lambda = 1/\sqrt{3}, \quad A = \begin{pmatrix} -1/\sqrt{3} & 0 & \\ 0 & -1/\sqrt{3} & \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix}.$$

The order of the eigenvalues on the diagonal is immaterial, for any permutation of the diagonal entries of A produces a point on the same orbit. Indeed, any such permutation is accomplished by the operation PAP^{-1}, where P is a permutation matrix, and such a P is an element of O(3). One of the orbits above gives the maximum of the functional $\frac{1}{3}\,\mathrm{tr} A^3$ on the sphere $\frac{1}{2}\mathrm{tr} A^2 = 1$, the other orbit is the minimum. The isotropy subgroup in each case is O(2) (rotations which leave $\binom{0}{0}{}_{1}$ invariant), so each extremal is axisymmetric.

The results above were obtained jointly with Professor L. Green. The method, while quite straightforward in the case $l=2$, becomes extremely complicated already in the case $l=4$ and so does not seem to be a practical approach to the resolution of the bifurcation equations (5.1) in the general case. It is interesting, nevertheless, to compare this approach with that of Michel and Radicati[7] in their work on symmetry breaking problems in physics.

They study the action of SU(n) on the vector space Q of Hermitian traceless matrices A with the inner product (5.7). It can be proved that there are two linearly independent trilinear invariants of this action, viz.,

$$\{A, B, C\} = \tfrac{1}{2}\sqrt{n}\ \mathrm{tr}(AB + BA)C, \quad [A, B, C] = -\tfrac{1}{2} i\,\mathrm{tr}[A, B]C$$

with $\{\ ,\ ,\ \}$ completely symmetric and $[\ ,\ ,\ ]$ completely antisymmetric. The bilinear form (5.7) is the only second order invariant. From this it can be concluded that there are only two linearly independent algebras on A with SU(n) as automorphism group. One is the Lie algebra whose multiplication law is

$$x \wedge y = -\tfrac{1}{2} i [x, y]$$

and the other is that with multiplication law

$$x \vee y = \tfrac{1}{2}\sqrt{n}\,(xy + yx) - (1/\sqrt{n})\,\mathrm{tr} xy.$$

Michel and Radicati are led to study the equation [(III.17), p. 194 of Ref. 7]

$$q \vee q + N(q) q = 0, \tag{5.10}$$

where $N(q)$ is a real number. Equation (5.10) is precisely equivalent to (5.8).

6. EXTREMAL METHODS AND STABILITY OF BIFURCATING SOLUTIONS

Having shown in the previous section that the reduced bifurcation equations sometimes possess a gradient structure as a consequence of their symmetry, we investigate in this section the relationship between the extremal properties of solutions and stability of the bifurcating solutions. Let us again assume v is a solution of the reduced bifurcation equations (5.1). The Jacobian of these equations at v is the linear operator

$$J_\sigma(v)x = \sigma x + 2B(x, x).$$

We know from Theorem 2.3 that the stability of the bifurcating solutions is determined by the eigenvalues of the linear operator $J_\sigma(v)$. Now suppose B is the gradient of the functional

$$\mathcal{J}(v) = \tfrac{1}{3}\langle B(v, v), v\rangle$$

and that v is the extremal of the variational problem

$$\max_{\langle v,v\rangle=1} \mathcal{J}(v). \tag{6.1}$$

We calculate the second variation of the variational problem at v. Let $x(t)$ be a curve on the unit sphere such that $x(0)=v$. Then, if $\mathcal{J}(v)$ attains a maximum at v,

$$\frac{d^2}{dt^2}\mathcal{J}(x(t)) = \langle B(v, v), \ddot{x}\rangle + 2\langle B(v, \dot{x}), \dot{x}\rangle \le 0$$

and

$$\frac{d^2}{dt^2}\frac{1}{2}\langle x, x\rangle = \langle v, \ddot{x}\rangle + \langle \dot{x}, \dot{x}\rangle = 0.$$

From (5.1), $\sigma v + B(v, v) = 0$, so

$$-\sigma\langle v, \ddot{x}\rangle + 2\langle B(v, \dot{x}), \dot{x}\rangle \le 0, \quad \langle \sigma\dot{x} + 2B(v, \dot{x}), \dot{x}\rangle \le 0$$

for all tangent vectors $\dot{x}$ and v. Consequently,

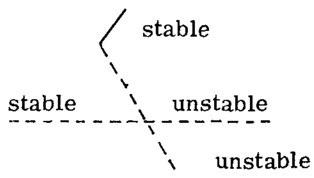
$$\langle J_\sigma(v)\dot{x}, \dot{x}\rangle \le 0 \tag{6.2}$$

for all tangent vectors $\dot{x}$. Furthermore, $J_\sigma(v)$ leaves the tangent plane at v invariant. In fact, the equation $\langle v, \dot{x}\rangle = 0$ describes the tangent plane at v and also $\langle B(v, \dot{x}), v\rangle = \langle B(v, v), \dot{x}\rangle = -\sigma\langle v, \dot{x}\rangle = 0$. Therefore, if $\langle v, \dot{x}\rangle = 0$, then $\langle J_\sigma(v)\dot{x}, v\rangle = 0$ as well, and the tangent plane at v is preserved. So $J_\sigma(v)$ maps the tangent plane to itself, and (6.2) tells us $J_\sigma(v)$ is negative semidefinite at a local maximum v. The normal vector to the tangent plane is v itself, and $J_\sigma(v)v = \sigma v + 2B(v, v) = B(v, v) = -\sigma v$. Therefore, the remaining eigenvalue of $J_\sigma(v)$ is $-\sigma$. Since at an extremal v

$$-\sigma = \langle B(v, v), v\rangle / \langle v, v\rangle,$$

the eigenvalue $-\sigma$ is positive at a positive maximum of $\langle B(v, v), v\rangle$. We have proved

Theorem 6.1: Suppose the reduced bifurcation equations (5.1) have a gradient structure and that a solution v is obtained as a maximum of the variational problem (5.2). Then one eigenvalue of the Jacobian $J_\sigma(v) = \sigma I + 2B(v)$ is positive and the rest are nonpositive. Accordingly, from Theorem 2.3 it follows that the corresponding branch of solutions, which in this case is transcritical, has one unstable subcritical mode:



This situation occurs often in bifurcation problems and is depicted schematically in the above figure. When the effect of higher order terms is included the subcritical branch may "bend back" and regain stability. Such a situation is called "hard excitation" in nonlinear oscillation theory or "snap through" instability in the case of buckling theory. The situation depicted in the figure of Theorem 6.1 can lead to sudden jump discontinuities and hysteresis effects as the parameter λ is varied in the vicinity of its critical value.

A similar analysis can be carried through the cubic case, when the reduced bifurcation equations take the form

$$\sigma x - B(x, x, x) = 0. \tag{6.3}$$

Again, if B is the gradient of the functional $\frac{1}{4}\langle B(x), x\rangle$ then equations (6.3) are the Euler-Lagrange equations for the variational problem

$$\min_{\langle x,x\rangle=1} \tfrac{1}{4}\langle B(x, x, x), x\rangle.$$

It can easily be shown, by the same analysis as before, that the eigenvalues of the Jacobian are always nonpositive at a positive minimum of the quartic $\langle B(x, x, x), x\rangle$ on the sphere $\langle x, x\rangle = 1$. Hence at a positive minimum (which does not necessarily exist) we get stable supercritically bifurcating solutions. The bifurcations are one-sided—supercritical positive extrema and subcritical at negative extrema. Subcritically bifurcating solutions are always unstable. (For further results see Sather.[9])

7. SPECIAL RESULTS

In this section we discuss the special results which can be obtained by direct calculations for low values of l: $l = 1, 2, 3, 4$.

For $l = 1$ the reduced bifurcation equations are

$$\begin{aligned} \sigma z_1 &= a z_1(z_0^2 - 2z_1 z_{-1}), \\ \sigma z_2 &= a z_0(z_0^2 - 2z_1 z_{-1}), \\ \sigma z_{-1} &= a z_{-1}(z_0^2 - 2z_1 z_{-1}), \end{aligned} \tag{7.1}$$

where the parameter a is to be considered a fixed real constant. For real solutions we require $\bar{z}_m = (-1)^m z_{-m}$, hence $z_0^2 - 2z_1 z_{-1} = z_0^2 + 2|z_i|^2$. A nontrivial solution of (6.1) must satisfy

$$z_0^2 + 2|z_1|^2 = \sigma/a, \tag{7.2}$$

from which we see that σ/a must be positive. Therefore, the bifurcation is supercritical ($\sigma > 0$) if $a > 0$ and subcritical if $a < 0$. The full set of solutions of (6.1) is

$$z_0 = \sqrt{\sigma/a}\cos\theta, \quad z_{\pm1} = \pm\sqrt{\sigma/2a}\sin\theta\exp(\pm i\varphi). \tag{7.3}$$

The eigenvalues of the Jacobian are constant on orbits and are most easily evaluated at $z_0 = \sqrt{\sigma/a}$, $z_{\pm1} = 0$; they are $0, 0, -2\lambda$, reflecting the fact that the orbit of solutions is two-dimensional. Since the Jacobian is not invertible, the implicit function theorem cannot be used to continue solutions of the reduced bifurcation equations to the full equations. One can *a fortiori* restrict oneself to a subspace of axisymmetric solutions where the problem reduces to bifurcation at a simple eigenvalue; but the question remains as to whether all solutions of the full bifurcation equations are obtained in this way. The case $l = 1$ arises in spherical convection problems when the inner and outer surfaces are free surfaces (Chossat,[19] p. 19).

Now let us turn to the case $l=2$, which was first treated in full by Busse[1] and in Sec. 5 of the present paper. The reduced bifurcation equations are (setting $\sigma/a=1$ without loss of generality)

$$\begin{aligned} z_2 &= (2\sqrt{2}/7 z_2 z_0 - \sqrt{3}/7 z_1^2), \\ z_1 &= (-2\sqrt{1}/14 z_1 z_0 + 2\sqrt{3}/7 z_2 z_{-1}), \\ z_0 &= (-\sqrt{2}/7 z_0^2 + 2\sqrt{1}/14 z_1 z_{-1} + 2\sqrt{2}/7 z_2 z_{-2}), \\ z_{-1} &= (-2\sqrt{1}/14 z_{-1} z_0 + 2\sqrt{3}/7 z_{-2} z_1), \\ z_{-2} &= (2\sqrt{2}/7 z_{-2} z_0 - \sqrt{3}/7 z_{-1}^2). \end{aligned} \tag{7.4}$$

This system of equations is already quite complicated (and is destined to get worse). However, one can obtain a special class of solutions by setting $z_{\pm1}=0$ and taking z_2 to be real. The equations then reduce to two equations in two unknowns (since $z_{-2}=z_2$), viz.,

$$z_2 = 2\sqrt{2}/7 z_2 z_0,$$
$$z_0 = -\sqrt{2}/7 z_0^2 + 2\sqrt{2}/7 z_2^2.$$

Two solution sets are

$$z_{\pm2} = \sqrt{21/16}, \quad z_{\pm1}=0, \quad z_0=\sqrt{7}/8 \tag{7.5}$$

and

$$z_{\pm2}=z_{\pm1}=0, \quad z_0=-\sqrt{7}/2. \tag{7.6}$$

One can evaluate the third order invariant $p_3(z)=\frac{1}{3}\sum F_m \overline{z_m}$ at the two solutions (properly normalized) and one finds $p_3=\sqrt{2}/7$ for the first solution and $-\sqrt{2}/7$ for the second. Thus these two special solutions lie on the maximum and minimum orbits of the functional p. The Jacobian of the reduced bifurcation equations can be calculated and its eigenvalues determined. We omit the details, but the eigenvalues in both cases are $[3,3,-1,0,0]$. Again, both orbits are axisymmetric. Due to the presence of the zero eigenvalues the implicit function theorem cannot be used directly here, but there is an alternative method. Let us restrict ourselves to solutions with the reflection symmetry property $z_{-m}=(-1)^m z_m$. Then all solutions have real values z_m. Under those conditions the last two equations of (6.4) are identical with the first two. More generally,

Theorem 7.1: Let the general bifurcation equations $F_m(z_{-l},\dots z_l)=0$ be restricted to the subclass of solutions with the symmetry property

$$z_m=(-1)^m z_{-m}. \tag{7.7}$$

Then $F_{-m}(z_{-l},\dots z_l)=(-1)^m F_m(z_{-l},\dots,z_l)$ and the bifurcation equations to $(l+1)$ real equations in $(l+1)$ real unknowns.

Proof: The reality condition $\overline{z_m}=(-1)^m z_{-m}$ and (7.7) imply that z_m is real. Furthermore, the bifurcation equations satisfy

$$\overline{F_m(z_{-l},\dots,z_l)}=(-1)^m F_{-m}(z_{-l},\dots,z_l), \tag{7.8}$$

$$\overline{F_m(z_{-l},\dots,z_l)}=F_m(\overline{z_{-l}},\dots,\overline{z_l}). \tag{7.9}$$

The property (7.8) following from the reality condition (3.6) and (7.9) being a consequence of the fact that all coupling coefficients are real. (They are obtainable by the Lie algebra methods outlined in Sec. 3.) Combining (7.7), (7.8), and (7.9), we obtain

$$\begin{aligned} F_{-m}(\cdots z_m \cdots) &= (-1)^m F_m(\cdots z_m \cdots) \\ &= (-1)^m F_m(\cdots z_m \cdots). \end{aligned}$$

Therefore, the last l equations coincide with the first l when z_{-m} is replaced by $(-1)^m z_{-m}$.

We omit the details, but if one computes the Jacobian of the first three equations of (7.4) at the special solutions (7.5) and (7.6) he obtains an invertible operator. Therefore, all solutions of the bifurcations can be obtained by solving the reduced bifurcation equations and applying the implicit function theorem when the restriction (7.7) is in force. A similar situation prevails in the case $l=4$. I conjecture that it is valid for all even l.

For $l=3$ there are two distinct covariant terms. They are obtained as follows. We must take F_3 to be

$$\begin{aligned} F_3 &= a z_3^2 z_{-3} + b z_3 z_2 z_{-2} + c z_3 z_1 z_{-1} \\ &\quad + d z_3 z_0^2 + e z_2 z_1 z_0 + f z_2^2 z_{-1} + g z_1^3. \end{aligned}$$

Applying the condition $J_+F_3=0$, we are led to the system of five equations in seven unknowns:

$$\begin{bmatrix} \beta_{-3} & \beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{-2} & \beta_1 & 0 & 0 & 2\beta_2 & 0 \\ 0 & 0 & \beta_{-1} & 2\beta_1 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & \beta_{-1} & 0 \\ 0 & 0 & 0 & 0 & \beta_0 & 0 & 3\beta_1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = 0,$$

where $\beta_m=\sqrt{(3-m)(3+m+1)}$.

One solution is obtained by taking $g=0$ and $d=1$. Then $e=f=0$ and we get

$$F_3 = z_3(z_0^2 - 2z_1 z_{-1} + 2z_2 z_{-2} - 2z_3 z_{-3}).$$

The quantity in parentheses is the second order invariant, and so is annihilated by the application of any of the J operators. Therefore, one mapping is

$$F_m = z_m(z_0^2 - 2z_1 z_{-1} + 2z_2 z_{-2} - 2z_3 z_{-3}).$$

A second choice is $g\neq0$, $d=0$. The choice $g=\sqrt{7}$ leads to

$$\begin{aligned} G_3 &= 9\sqrt{60}/7 z_3^2 z_{-3} - 9\sqrt{60}/7 z_3 z_2 z_{-2} \\ &\quad + 3\sqrt{60}\, z_3 z_1 z_{-1} - 3\sqrt{10}\, z_2 z_1 z_0 \\ &\quad + (30/\sqrt{7}) z_2^2 z_{-1} + \sqrt{7}\, z_1^3. \end{aligned}$$

The lower weight polynomials are obtained by successively applying the lowering operator J_-.

The general reduced bifurcation equations in this case take the form

$$\lambda z_m = A F_m + B G_m,$$

where the parameters A and B depend on the external physical parameters of the problem. Such a situation occurs in the Bénard problem and gives rise to mechanisms for pattern selection.[14]

$l=4$: The quadratic terms in this case are

$$F_4=(1/\sqrt{5})z_4z_0-(1/\sqrt{2})z_3z_1+(3/2\sqrt{14})z_2^2,$$
$$F_3=(1/\sqrt{2})z_4z_{-1}-(3/2\sqrt{5})z_3z_0+(1/\sqrt{14})z_2z_1,$$
$$F_2=(3/\sqrt{14})z_4z_{-2}-(1/\sqrt{14})z_3z_{-1}-(11/14\sqrt{5})z_2z_0+(3/7\sqrt{2})z_1^2,$$
$$F_1=(1/\sqrt{2})z_4z_{-3}+(1/\sqrt{14})z_3z_{-2}-(6/7\sqrt{2})z_2z_{-1}+(9/7\sqrt{20})z_1z_0,$$
$$F_0=(1/\sqrt{5})z_4z_{-4}+(3/2\sqrt{5})z_3z_{-3}-(11/14\sqrt{5})z_2z_{-2}-(9/14\sqrt{5})z_1z_{-1}+(9/14\sqrt{5})z_0^2.$$

The remaining polynomials are found from those above by the relationship $F_{-m}(z_{-4},\ldots,z_{+4})=(-1)^mF_m(z_{-4},\ldots,z_4)=(-1)^mF_m(\cdots(-1)^mz_{-m}\cdots)$. There are many possible solutions to the bifurcation equations in this case. Busse has found two special solutions:

(1) Axisymmetric solutions: $z_{\pm1}=\cdots=z_{\pm4}=0$, $z_0\neq 0$; and (2) octahedral solutions: $z_4=z_{-4}=5/\sqrt{14}$, $z_0=\sqrt{5}$, $z_{\pm1}=z_{\pm2}=z_{\pm3}=0$. Busse conjectures, on the basis of numerical work, that the second solution is the one which maximizes the third order invariant. An analysis of the Jacobian shows that the axisymmetric solution is a saddle point and that the octahedral solution is a candidate for the maximum.

The Jacobian of the reduced bifurcation equations (we set $\sigma=1$ and drop the normalization condition $|z|=1$; the results are affected only by a possible change of scale) at the special solution $z_0=\sqrt{5}$, $z_4=z_{-4}=5/\sqrt{14}$, $z_{\pm1}=z_{\pm2}=z_{\pm3}=0$ is

$$J-I=\begin{bmatrix} 0&0&0&0&\sqrt{5}/14&0&0&0&0\\ 0&-5/2&0&0&0&5/\sqrt{7}&0&0&0\\ 0&0&-25/14&0&0&0&15/14&0&0\\ 0&0&0&-5/14&0&0&0&5/2\sqrt{7}&0\\ \sqrt{5}/14&0&0&0&2/7&0&0&0&\sqrt{5}/14\\ 0&5/2\sqrt{7}&0&0&0&-5/14&0&0&0\\ 0&0&15/14&0&0&0&-25/14&0&0\\ 0&0&0&5/2\sqrt{7}&0&0&0&-5/2&0\\ 0&0&0&0&\sqrt{5}/14&0&0&0&0 \end{bmatrix}.$$

The eigenvalues of this matrix can be determined by restricting the matrix to certain invariant subspaces (determined by inspection), as follows. Let $\mathbf{e}_i$ denote the column vector with a 1 in the ith row and zeros everywhere else. The subspaces $\{a\mathbf{e}_3+b\mathbf{e}_7\}$, $\{\mathbf{e}_1+\mathbf{e}_9\}$, $\{a\mathbf{e}_1+b\mathbf{e}_5+c\mathbf{e}_9\}$, $\{a\mathbf{e}_2+b\mathbf{e}_6\}$, $\{a\mathbf{e}_4+b\mathbf{e}_8\}$ are all invariant, and one has to calculate the eigenvalues at most of a 3×3 matrix. The complete set of eigenvalues is

$$\{0,0,0,-20/7,\ -20/7,\ -20/7,\ -5/7,\ -5/7,1\}.$$

Since only one eigenvalue is positive, this octahedral solution is a possible candidate for the maximum of the extremal problem. The axisymmetric solution above, however, is definitely a saddle point of the variational problem; the eigenvalues of the Jacobian are

$$\{0,0,20/9,\ 20/9,\ 10/3,\ 10/3,\ -5/9,\ -5/9,\ -1\}.$$

Busse's article also contains a discussion of the situation for higher values of l, and special solutions are given for $l=6,8$. His special solutions belong to one of two classes (besides the axisymmetric solutions):

$$z_0\neq0,\ z_n,\ z_{2n}\neq0,\ \tfrac{1}{3}l<n\leq\tfrac{1}{2}l, \tag{7.10a}$$

$z_m=0$ otherwise;

$$z_0\neq0,\ z_n\neq0 \text{ for a single } n>l/2, \tag{7.10b}$$

$z_m=0$ otherwise.

The axisymmetric solutions never give a maximum except in the case $l=2$.

8. APPLICATIONS

Convection problems in spherical geometries arise naturally in geophysical problems and have been discussed by many authors (Chandrasekar,[10] Busse,[1] Chossat[19]). Convective phenomena in fluid media are generally modeled by the Boussinesq equations, which, in dimensionless variables, take the form

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=\Delta\mathbf{u}-\nabla p+\lambda g_1(r)\theta\mathbf{r}+\epsilon\omega\mathbf{u}\times\hat{k}, \tag{8.1}$$

$$\frac{\partial\theta}{\partial t}=\frac{1}{\Pr}(\Delta\theta+\lambda\beta_1(r)\mathbf{u}\cdot\mathbf{r})-\mathbf{u}\cdot\nabla\theta,$$

$$\operatorname{div}\mathbf{u}=0,$$

where $\mathbf{u}$ is the fluid velocity field, θ is the temperature perturbation, p is the hydrodynamic pressure, and $\mathbf{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is the position vector. Pr is the Prandtl number and λ is the Rayleigh number; $g_1(r)$ is the gravitational field and $\beta_1(r)$ is the steady state temperature gradient. The term $\omega\mathbf{u}\times\hat{k}$ is the coriolis term due to rotation of the fluid. The operator $\mathbf{u}\times\hat{k}$ breaks O(3) symmetry, as it is only invariant under rotations about the $\hat{k}$ axis.

In geophysical applications these equations are considered on a spherical shell $\mathcal{N}<r<1$ with appropriate boundary conditions. When both surfaces are free, the kernel of the linearized equations contains the space V^1 (which transforms as D^1) (Chossat,[19] p. 19). When both surfaces are rigid and $\mathcal{N}$ is in the vicinity of 0.3 the kernel of L_λ for the critical value of λ_c transforms as D^2; but, as $\mathcal{N}\to1$, the kernel of L_{λ_c} transforms as D^l for higher and higher values of l (Chossat, personal communication). Chossat's thesis contains an extensive discussion of the linearized eigenvalue problem for the

Boussinesq equations (7.1) in a spherical shell, and also discusses the effect of the symmetry breaking term $\omega u \times k$ on the bifurcation point. Depending on the sign of ω, one gets either a bifurcation of stationary solutions or time periodic solutions.

The buckling of perfectly spherical shells has also been the subject of much investigation. (See esp. Sather.[9,12]) Many of the investigations have been limited to Axisymmetric buckling, as in Bauer, Keller and Reiss.[11] This restriction is certainly justified if $\ker G_u(0,0)$ transforms as D^l for $l=1,2$; but already in the case $l=4$ Busse's result shows that the axisymmetric solutions are generally not the relevant bifurcating solutions. If the equations of elasticity exhibit the same behavior as is supposed for the convection equations—that is, if $\ker G_u(0,0)$ transforms D^l for higher and higher l as $N \to 1$—then the bifurcation problem for higher values of l is also of interest in buckling problems. Actually, experiments on the buckling of very thin spherical shells indicates that this is precisely the case.

Finally, the symmetry breaking problems studied by Michel and Radicati[7,8] also lead directly to the analog of the reduced bifurcation equations (5.1) but with SO(3) [or SU(2)] replaced by SU(3).

APPENDIX

Let us derive the expression (4.3) for the generating function for the characters $\chi_n(g)$ of $(\Gamma^{\otimes n})s$. Fix the group element g and let the eigenvectors of Γ on V be $\mathbf{e}_1, \ldots, \mathbf{e}_r$ with eigenvalues $\lambda_1, \ldots, \lambda_s$. The vector space $(V^{\otimes n})s$ is spanned by the vectors $(\mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_r})s$, which we may represent as

$$\mathbf{w} = \sum_{\sigma \in s_r} \mathbf{e}_{i_{\sigma(1)}} \otimes \cdots \otimes \mathbf{e}_{i_{\sigma(r)}}.$$

The action of $\Gamma^{\otimes n}$ on $\mathbf{w}$ is simply $\Gamma^{\otimes n}\mathbf{w} = \lambda_1^{m_1} \cdots \lambda_r^{m_r} w$, where $m_1 + \cdots + m_r = n$. Thus a vector in $(V^{\otimes n})s$ may be represented by its occupation numbers $m_1, \ldots, m_r$ (where m_1 = times $\mathbf{e}_1$ occurs, and so forth). The trace of $\Gamma^{\otimes n}$ is therefore

$$\chi_{(n)}(g) = \operatorname{tr} \Gamma^{\otimes n}(g) = \sum_{m_1 + \cdots + m_r = n} \lambda_1^{m_1} \cdots \lambda_r^{m_r}.$$

Multiplying by z^n and summing, we get

$$\sum_{n=0}^{\infty} z^n \chi_{(n)}(g) = \sum_{n=0}^{\infty} \sum_{m_1 + \cdots + m_r = n} (z\lambda_1)^{m_1} \cdots (z\lambda_r)^{m_r}$$

$$= \sum_{m_1, \ldots, m_r = 0}^{\infty} (z\lambda_1)^{m_1} \cdots (z\lambda_r)^{m_r}$$

$$= \prod_{i=1}^{r} \frac{1}{(1 - z\lambda_i)} = \det(I - z\Gamma_{(g)})^{-1}.$$

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Systems of differential inequalities and stochastic differential equations. IV[a)]

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Consider the system of stochastic functional differential equations

$$x'(t,\omega)=f(t,x(t,\omega),\ x_t(\omega),\omega),\ x_{t0}(\omega)=\phi_0(\omega),$$

where $f(t,x(t,\omega),\ x_t(\omega),\omega)$ is a product measurable n-dimensional random vector functional whenever $x(t,\omega)$ is a product measurable random function, and it satisfies the desired regularity conditions to assure the existence of solution process. By developing systems of random differential inequalities, very general comparison theorems in the framework of a vector Lyapunov function are obtained, and, furthermore, sufficient conditions are given for the stability of solutions in probability, in the mean and with probability one.

1. INTRODUCTION

The stability analysis of stochastic functional differential systems has been the subject of many investigations.[1–10] Most of the stability analysis of stochastic functional differential system is centered around the stability analysis of stochastic functional differential systems of Ito type[2,6,7] and functional differential sytems with Markov coefficients[1–4,9,10] in the context of single or scalar Lyapunov functionals or functions. However, the stability analysis of stochastic functional differential systems with nonwhite excitations is remained unattempted.

Very recently, by developing very general compari-theorems for Itô type stochastic ordinary[11,12] and functional[8,13] differential systems, and for ordinary[14] and functional[4,9] differential systems with Markov coefficients in the context of deterministic differential inequalities, and for stochastic ordinary differential systems with nonwhite noise coefficients[15] in the context of random differential inequalities, sufficient conditions are given for stability and boundedness of these stochastic differential systems. Moreover, it has been demonstrated that the concept of vector Lyapunov functions and the theory of differential inequalities are promising tools for undertaking the stability analysis of deterministic nonhereditary,[16] deterministic hereditary,[17] and random nonhereditary[18] competitive-cooperative processes in biological, physical, and social sciences.

In this paper, we initiate the stability analysis of stochastic functional differential systems with random coefficients and random delay. We develop the theory of systems of random functional differential inequalities, and obtain a very general comparison theorem in the framework of a random vector Lyapunov function and systems of random functional differential inequalities.

The paper is organized as follows:

In Sec. 2, depending on the convergence concepts in probabilistic analysis, we define various notions of stability. In Sec. 3, we develop the theory of systems of random functional differential inequalities. The obtained results extend the deterministic[19,20] and random non-hereditary[15] results to random hereditary. In Sec. 4, by employing the concept of random vector Lyapunov function, a very general comparison theorem in the context of the systems of random functional differential inequalities is developed. Furthermore, a general comparison theorem that is based on a random vector Lyapunov function and a minimal class of functions is also developed. All of these results extend the deterministic[19,21] and random nonhereditary[15] results to stochastic hereditary. In Sec. 5, we apply the comparison theorems that are developed in Sec. 4, to study stability analysis of random functional differential systems. Finally, in Sec. 6, some examples are given to show the usefulness of our results.

2. NOTATIONS AND DEFINITIONS

Let R^n denote the n-dimensional Euclidean space with a convenient norm $\|\cdot\|$. We also denote by the same symbol $\|\cdot\|$ the corresponding norm of a matrix. Let R_+ denote the nonnegative real line while R will be used for real line. Let $(\Omega,\mathcal{J},P)$ be a complete probability space. Let $S(R^n)$ denote the set of random vectors defined on $(\Omega,\mathcal{J},P)$ into R^n. For $x\in S(R^n)$, the qth moment of x is defined by $E(\|x\|^q)=\int_\Omega\|x(\omega)\|^q P(d\omega)$, $0<q<\infty$. For $0<\rho\leq\infty$, $D=D(0,\rho,R^n)=\{x\in R^n\colon \|x\|<\rho\}$, and $D(S(R^n))=D(0,\rho,S(R^n))=\{x\in S(R^n)\colon\|x(\omega)\|<\rho$ with probability one$\}$. Given $\tau>0$, let $\mathcal{C}^n=C[[-\tau,0],R^n]$ denote the space of continuous functions defined on $[-\tau,0]$ into R^n, and let $S(\mathcal{C}^n)=C[[-\tau,0],S(R^n)]$ denote the space of almost sure sample continuous random functions with domain $[-\tau,0]$ and range in R^n. For $\phi\in\mathcal{C}^n$, we define $\|\phi\|_0=\sup_{-\tau\leq s\leq 0}\|\phi(s)\|$. Let $(\cdot)^T$ stand for the transpose of a vector or a matrix. Suppose that $C[[-\tau,\infty),R^n]$. For $t\geq 0$, we shall let x_t denote the translation of the restriction of x to the interval $[t-\tau,t]$; more specifically, x_t is an element of $\mathcal{C}^n$ defined by $x_t(s)=x(t+s)$, $-\tau\leq s\leq 0$. $C[[-\tau,\infty),S(R^n)]$ can be defined, similarly, and $x_t(\omega)\in S(\mathcal{C}^n)$ is defined by $x_t(\omega)=x(t+s,\omega)$, $-\tau\leq s\leq 0$. $\mathcal{C}^n_+=C[[-\tau,0],$

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$R_+^n]$, $S(C_+^n)$, $C=C(0,\rho)=\{\phi\in C^n: \|\phi\|_0<\rho\}$ and $S(C)=S(C(0,\rho))=\{\phi\in S(C^n): \|\phi\|_0<\rho$ with probability one$\}$. $AC[R_+, S(R^n)]$ denote the set of all almost surely absolutely sample continuous random functions defined on R_+ into R^n. We shall mean by $M[R_+\times D\times C, S(R^n)]$ the class of random functionals $f(t,x,x_t,\omega)$ defined on $R_+\times D\times C$ into R^n such that $f(t,x(t,\omega),x_t(\omega),\omega)$ is product measurable whenever $x(t,\omega)$ is product measurable.

Consider the system of stochastic functional differential equations of the type

$$x'(t,\omega)=f(t,x(t,\omega),x_t(\omega),\omega),\ x_{t_0}(\omega)=\phi_0(\omega), \qquad (2.1)$$

where $f\in M[R_+\times D\times C, S(R^n)]$, and f is smooth enough to guarantee the existence of a sample solution process $x(t,\omega)=x(t_0, \phi_0)(t,\omega)$ of (2.1) for $t>t_0$. For existence and uniqueness theorems, see Refs. 1 and 2.

We shall assume that $f(t,0,0,\omega)\equiv 0$ with probability one, so that the system (2.1) possesses the trivial solution process $x(t,\omega)\equiv 0$ with probability one (w.p.1).

Depending on the mode of convergence in the probabilistic analysis and stability definitions for ordinary stochastic differential systems,[15] we shall formulate some definitions of stability.

Definition 2.1: The trivial solution of (2.1) is said to be:

(SP_1) *stable in probability*, if for each $\epsilon>0$, $\eta>0$, $t_0\in R_+$, there exists a $\delta=\delta(t_0,\epsilon,\eta)>0$ such that

$$P\{\omega: \sup_{-\tau\le s\le 0}\|\phi_0(s,\omega)\|>\delta\}<\eta$$

implies

$$P\{\omega: \|x(t_0,\phi_0)(t,\omega)\|\ge\epsilon\}<\eta,\quad t\ge t_0;$$

(SP_2) *asymtotically in probability*, if it is stable in probability and, if for any $\epsilon>0$, $\eta>0$, $t_0\in R_+$, there exist $\delta_0=\delta_0(t_0)>0$ and $0<T=T(t_0,\epsilon,\eta)$ such that

$$P\{\omega: \sup_{-\tau\le s\le 0}\|\phi_0(s,\omega)\|>\delta_0\}<\eta$$

implies

$$P\{\omega: \|x(t_0,\phi_0)(t,\omega)\|\ge\epsilon\}<\eta,\quad t\ge t_0+T;$$

(SM_1) *stable in the mean*, if for each $\epsilon>0$, $t_0\in R_+$, there exists a $0<\delta=\delta(t_0,\epsilon)$ such that the inequality

$$\sup_{-\tau\le s\le 0} E[\|\phi_0(s,\omega)\|]\le\delta$$

implies

$$E[\|x(t_0,\phi_0)(t,\omega)\|]<\epsilon,\quad t\ge t_0;$$

(SM_2) *asymptotically stable in the mean*, if it is stable in the mean and, if for any $\epsilon>0$, $t_0\in R_+$, there exist positive numbers $\delta_0=\delta_0'(t_0)$ and $T=T(t_0,\epsilon)$ such that the inequality

$$\sup_{-\tau\le s\le 0} E[\|\phi_0(s,\omega)\|]\le\delta_0$$

implies

$$E[\|x(t_0,\phi_0)(t,\omega)\|]<\epsilon,\quad t\ge t_0+T;$$

(SS_1) *stable with probability one* (or almost surely sample stable), if for $\epsilon>0$, $t_0\in R_+$, there exists a positive number $\delta=\delta(t_0,\epsilon)$ such that the inequality

$$\sup_{-\tau\le s\le 0}\|\phi_0(s,\omega)\|\le\delta\ \text{ w.p.1}$$

implies

$$\|x(t_0,\phi_0)(t,\omega)\|<\epsilon,\quad t\ge t_0 \text{ w.p.1};$$

(SS_2) *asymptotically stable with probability one* (or almost surely sample asymptotically stable), if it is stable with probability one and, if for any $\epsilon>0$, $t_0\in R_+$, there exist positive numbers $\delta_0=\delta_0(t_0)$ and $T=T(t_0,\epsilon)$ such that the inequality

$$\sup_{-\tau\le s\le 0}\|\phi_0(s,\omega)\|\le\delta_0\ \text{ w.p.1}$$

implies

$$\|x(t_0,\phi_0)(t,\omega)\|<\epsilon,\quad t\ge t_0+T\ \text{ w.p.1}.$$

Definition 2.2: The trivial solution of (2.1) is said to be:

(USP_1) uniformly stable in probability, (USM_1) uniformly stable in the mean, and (USS_1) uniformly stable with probability one, if the δ's in Definition 2.1, (SP_1), (SM_1), and (SS_1), are independent of t_0, respectively;

(USP_2) uniformly asymptotically stable in probability, (USM_2) uniformly asymptotically stable in the mean, and (USS_2) uniformly asymptotically stable w.p.1, if (SP_1), (SM_1) and (SS_1) hold, and the corresponding δ's and T's in Definition 2.1, (SP_2), (SM_2), and (SS_2), are independent of t_0, respectively.

Based on Defintions 2.1 and 2.2, one can formulate other definitions of stability and boundedness,[11,12] analogously.

Consider now the following stochastic auxiliary functional and ordinary differential systems:

$$u'(t,\omega)=g(t,u(t,\omega),u_t(\omega),\omega),\ u_{t_0}(\omega)=\sigma_0(\omega) \qquad (2.2)$$

and

$$u'(t,\omega)=g(t,u(t,\omega),\omega),\ u(t_0,\omega)=u_0(\omega), \qquad (2.3)$$

respectively, where g in (2.2) belongs to $LC[R_+\times R^m\times C^m, S(R^m)]$, $LC[R_+\times R^m\times C^m, S(R^m)]$ stands for the class of random functionals $g(t,\sigma(0),\sigma,\omega)$ defined on $R_+\times R^m\times C^m$ into R^m such that $g(t,\sigma(0),\sigma,\omega)$ satisfies the Caratheodory condition in $(t,\sigma(0),\sigma)$ for all most all $\omega\in\Omega$, i.e., $g(t,\sigma(0),\sigma,\omega)$ is continuous in $(\sigma(0),\sigma)$ for each $t\in R_+$ and Lebesgue measurable in t for each fixed $(\sigma(0),\sigma)$, with probability one, and there exists a product measurable random function $K: R_+\times\Omega\to R_+$ which is summable on R_+ w.p.1 such that $\|g(t,\sigma(0),\sigma,\omega)\|\le K(t,\omega)$ for $\sigma\in C(0,\rho)$, $0<\rho<\infty$ w.p.1; $g(t,\sigma(0),\sigma,\omega)$ is quasimonotone nondecreasing in $\sigma(0)$ and nondecreasing in σ for each $t\in R_+$ w.p.1; g in (2.3) belongs to $LC[R_+\times R^m, S(R^m)]$, and g satisfies the Caratheodory condition in (t,u) w.p.1; $g(t,u,\omega)$ is quasimonotone nondecreasing in u for each $t\in R_+$ w.p.1. Under these conditions, existence of maximal and minimal solutions w.p.1 can be shown analogous to the deterministic case[21] with simple modifications. Let $u(t,\omega)=u(t_0,\sigma_0)(t,\omega)$ and $u(t,\omega)=u(t,t_0,u_0,\omega)$ be any solutions of (2.2) and (2.3), respectively.

Relative to auxiliary differential systems (2.2) and (2.3), we need to define the corresponding stability definitions in our discussion, that may be defined, analogously. For example, we state the definition of stable in probability (SP_1^*) with respect to (2.2) and (2.3), respectively.

Definition 2.3: The trivial solution of (2.2) is said to be *stable in probability*, if given $\epsilon > 0$, $\eta > 0$, $t_0 \in R_+$, there exists a positive number $\delta = \delta(t_0, \epsilon, \eta)$ such that

$$P\{\omega: \sup_{-\tau \le s \le 0} \sum_{i=1}^{m} \sigma_{0i}(s,\omega) > \delta\} < \eta$$

implies

$$P\{\omega: \sum_{i=1}^{m} u_i(t_0,\sigma)(t,\omega) \ge \epsilon\} < \eta, \quad t \ge t_0.$$

Definition 2.4: The trivial solution of (2.3) is said to be *stable in probability*, if given $\epsilon > 0$, $\eta > 0$, $t_0 \in R_+$, there exists $\delta = \delta(t_0, \epsilon, \eta)$ such that

$$P\{\omega: \sum_{i=1}^{m} u_{i0}(\omega) > \delta\} < \eta$$

implies

$$P\{\omega: \sum_{i=1}^{m} u_i(t,\omega) \ge \epsilon\} < \eta, \quad t \ge t_0.$$

Definition 2.5: A function $b(r)$ is said to belong to the class K if $b \in C[R_+, R_+]$, $b(0) = 0$, $b(r)$ is strictly increasing in r.

Definition 2.6: A function $b(r)$ is said to belong to the class VK if $b \in C[R_+, R_+]$, $b(0) = 0$, $b(r)$ is convex and strictly increasing in r.

Definition 2.7: A function $a(t,r)$ is said to belong to the CK if $a \in C[R_+ \times R_+, R_+]$, $a(t,0) \equiv 0$, and $a(t,r)$ is concave and increasing in r for each fixed $t \in R_+$.

Definition 2.8: Let G be a function defined on R^n into R^m. The function G is said to convex if each component G_i of G is convex for $1 \le i \le m$, and G is said to be concave if $-G_i$ is convex.

In order to avoid monotonicity, hereafter, it will be understood unless otherwise specified, that all equalities, inequalities, and relations that involve the random processes will hold with probability one.

3. RANDOM FUNCTIONAL DIFFERENTIAL INEQUALITIES

In this section, we shall establish the result that will be widely useful in the qualitative analysis of random functional differential systems of the type (2.1).

Theorem 3.1: Assume that:

(i) $g \in LC[R_+ \times R^m \times \mathcal{C}^m, S(R^m)]$, $g(t,\sigma(0),\sigma,\omega)$ is quasimonotone nondecreasing in $\sigma(0)$ and nondecreasing in σ for each fixed $t \in R_+$ w.p.1, and $r(t,\omega) = r(t_0,\sigma_0)(t,\omega)$ is the maximal solution process of the system of random functional differential equations (2.2) existing for $t \ge t_0$ w.p.1;

(ii) $m \in AC[[-\tau,\infty), S(R^m)]$, and $m(t,\omega)$ satisfies the inequality

$$m'(t,\omega) \le g(t, m(t,\omega), m_t(\omega), \omega) \tag{3.1}$$

almost everywhere on $(t,\omega) \in (t_0,\infty) \times \Omega$.

Then,

$$m_{t_0}(\omega) \le \sigma_0(\omega) \text{ w.p.1} \tag{3.2}$$

implies

$$m(t,\omega) \le r(t_0,\sigma_0)(t,\omega), \quad t \ge t_0 \quad \text{w.p.1}. \tag{3.3}$$

Proof: For any $i \in I = 1, 2, \ldots, m$, we define a functional

$$\bar{g}_i(t,\sigma,(0),\sigma,\omega) = g_i(t,\bar{\sigma}(0),\bar{\sigma},\omega), \tag{3.4}$$

where $\sigma \in \mathcal{C}^m$, $m_t(\omega) \le \bar{\sigma}(\omega)$, and for $j \in I$, $s \in [-\tau, 0]$

$$\bar{\sigma}_j(s,\omega) = \begin{cases} \sigma_j(s,\omega), & \text{if } m_j(s,\omega) \le \sigma_j(s), \\ m_j(s,\omega), & \text{if } m_j(s,\omega) > \sigma_j(s). \end{cases} \tag{3.5}$$

It is easy to see that $\bar{g}(t,\sigma(0),\sigma,\omega) \in LC[R_+ R^m \times \mathcal{C}^m, S(R^m)]$ and $\bar{g}(t,\sigma(0),\sigma,\omega)$ is quasimonotone nondecreasing in $\sigma(0)$ and nondecreasing in σ for each $t \in R_+$ w.p.1. Now, the rest of the proof follows by following the proof of Theorem 3.1 in Ref. 15 with slight modifications.

Remark 3.1: Theorem 3.1 is analogous to the deterministic Corollary 4 in Ref. 20 in the context of Remark 3 in Ref. 20. Furthermore, it is a direct extension of Theorem 3.1 in Ref. 15.

Remark 3.2: If, in Theorem 3.1, the inequalities (3.1) and (3.2) are reversed, then the conclusion (3.3) is to be replaced by

$$m(t,\omega) \ge \rho(t_0,\sigma_0)(t,\omega), \quad t \ge t_0 \quad \text{w.p.1},$$

where $\rho(t_0, \sigma_0)(t,\omega)$ is the minimal solution process of (2.2) existing for $t \ge t_0$ w.p.1.

4. COMPARISON THEOREMS

In this section by employing the concept of random vector function,[15] the systems of ordinary random differential inequalities[15] and the systems of random functional differential inequalities, we develop comparison theorems for the system of random functional differential equations (2.1). These results are not only useful in studying the qualitative properties of (2.1), but also useful for obtaining qualitative information of hereditary competitive-cooperative processes in biological, physical, medical, and social sciences, under random environmental as well as structural perturbations.

Let the function $V \in L[[-\tau,\infty) \times D, S(R^m)]$, where $L[[-\tau,\infty) \times D, S(R^m)]$ stands for the collection of random functions $V(t,x,\omega)$ defined on $[-\tau,\infty) \times D \times \Omega$ into R^m such that $V(t,x,\omega)$ is locally Lipschitzian in $(t,x) \in R_+ \times D$ w.p.1. We define a vector

$$D^+_{(2.1)} V(t,\phi(0),\phi,\omega) = \lim_{h \to 0^+} \sup \frac{1}{h} [V(t+h, \phi(0) + hf(t,\phi(0),\phi,\omega),\omega) - V(t,\phi(0),\omega)] \tag{4.1}$$

for $(t,\phi) \in R_+ \times C$. We note that, because of the assumptions on V, $D^+_{(2.1)} V(t,\phi(0),\phi,\omega)$ is a product measurable random vector.

Here and after, we shall assume that g in (2.2), g in (2.3), and the functions V and $D^+_{(2.1)} V(t,\phi(0),\phi,\omega)$ satisfy the following hypotheses:

(H_1) $g \in LC[R_+ \times R^m \times \mathcal{C}^m, S(R^m)]$, $g(t,\sigma(0),\sigma,\omega)$ is quasimonotone nondecreasing in $\sigma(0)$ and nondecreasing in σ for each fixed $t \in R_+$ w.p.1.

(H_2) Let $r(t,\omega) = r(t_0,\sigma_0)(t,\omega)$ be the maximal solution process of the auxiliary random functional differential system (2.2) existing for $t \geq t_0$.

(H_3) Assume that $g(t,0,0,\omega) \equiv 0$ almost everywhere (a.e.) on $R_+ \times \Omega$.

(H_4) $V \in L[[-\tau,\infty) \times D, S(R^m)]$, and for each $(t,\phi) \in R_+ \times C$, it satisfies the relation

$$D^+_{(2.1)} V(t,\phi(0),\phi,\omega) \leq g(t, V(t,\phi(0),\omega), V_t(\omega),\omega) \quad (4.2)$$

w.p.1, where $V_t(\omega) = V(t+s,\phi(s),\omega)$ for $s \in [-\tau, 0]$.

(H_5) Assume that the hypothesis (H_4) holds except that the inequality (4.2) is replaced by

$$A(t,\omega)D^+_{(2.1)}V(t,\phi(0),\phi,\omega) + A'(t,\omega)V(t,\phi(0),\omega) \leq g(t, A(t,\omega)V(t,\phi(0),\omega), (AV)_t(\omega),\omega), \quad (4.3)$$

where $A(t,\omega)$ is $m \times m$ random matrix function whose elements $a_{ij}(t,\omega)$ belong to $L[[-\tau,\infty), S(R_+)]$; $A^{-1}(t,\omega)$ exists, w.p.1, and $A^{-1}(t,\omega)A'(t,\omega)$ is product measurable and its off-diagonal elements are nonpositive w.p.1 on R_+; $(AV)_t(\omega) = A(t+s,\omega)V(t+s,\phi(s),\omega)$ for $s \in [-\tau,0]$.

(H_6) $g \in C[R_+ \times R^m_+, S(R^m_+)]$, $g(t,u,\omega)$ is quasimonotone nondecreasing in u for each fixed $t \in R_+$ w.p.1, and it satisfies the relation $\|g(t,u,\omega)\| \leq K(t,\omega)$, where $K(t,\omega)$ is a sample continuous random function.

(H_7) Let $r(t,\omega) = r(t,t_0,u_0,\omega)$ be the maximal solution process of the auxiliary random ordinary differential system (2.3) existing for $t \geq t_0$.

(H_8) Assume that $g(t,0,\omega) \equiv 0$ on $R_+ \times \Omega$ w.p.1.

(H_9) $V \in L[[-\tau,\infty) \times D, S(R^m_+)]$ and for each $i \in I$ and $(t,\phi) \in R_+ \times C$ such that $\phi \in \Omega_{iA}$, it satisfies the relation

$$A_i(t,\omega)D^+_{(2.1)}V(t,\phi(0),\phi,\omega) + A_i'(t,\omega)V(t,\phi(0),\omega) \leq g_i(t, A(t,\omega)V(t,\phi(0),\omega),\omega) \quad (4.4)$$

w.p.1, where $A(t,\omega)$ is as defined in (H_5); Ω_{iA} is defined by

$$\Omega_{iA} = \{\phi \in C: |(A_iV)_t(\omega)|_0 = A_i(t,\omega)V(t,\phi(0),\omega)\},$$

and $A_i(t,\omega)$ denotes the ith row of the random matrix $A(t,\omega)$.

(H_{10}) For $(t,x) \in [-\tau,\infty) \times D$

$$b(\|x\|) \leq \sum_{i \in I} V_i(t,x,\omega) \leq a(t,\|x\|), \quad (4.5)$$

where b, $a(t,\cdot) \in \mathcal{K}$.

(H_{11}) For $(t,x) \in [-\tau,\infty) \times D$, (4.5) holds with $b \in VK$, $a \in CK$.

(H_{12}) In addition to hypothesis (H_{10}), we assume that $a(t,r) = a(r)$.

(H_{13}) Assume that (H_{11}) holds with $a(t,r) = a(r)$.

On the basis of the result developed in the preceding section, we now prove the following comparison theorem which plays an important role in the qualitative analysis of solutions of random functional differential systems.

Theorem 4.1: Let the hypotheses (H_1), (H_2), and (H_4) be satisfied. Further assume that for any sample solution process $x(t_0,\phi_0)(t,\omega) = x(t,\omega)$ of (2.1) with $\phi_0 \in S(C)$ and

$$V_{t_0}(\omega) \leq \sigma_0(\omega). \quad (4.6)$$

Then

$$V(t, x(t_0,\phi_0)(t,\omega),\omega) \leq r(t_0,\sigma_0)(t,\omega) \quad (4.7)$$

as long as $x(t_0,\phi_0)(t,\omega) \in D(S(R^n))$ w.p.1 for $t \geq t_0$.

Proof: Let $x(t_0,\phi_0)(t,\omega)$ be any solution of (2.1) satisfying (4.6), and $x_t(t_0,\phi_0)(\omega) \in S(C)$ w.p.1 as long as $x(t_0,\phi_0)(t,\omega) \in D(S(R^n))$ w.p.1 for $t \geq t_0$. Set $\phi(\omega) = x_t(t_0,\phi_0)(\omega)$, which implies that $\phi(0,\omega) = x(t_0,\phi_0)(t,\omega)$. Define $m(t,\omega) = V(t,x(t,\omega),\omega)$ so that $m_t(\omega) = V(t+s, \phi(s),\omega)$. Since (4.6) holds, we have $m_{t_0}(\omega) \leq \sigma_0(\omega)$. For sufficiently small $h > 0$, we have

$$\begin{aligned} m(t+h,\omega) - m(t,\omega) &= V(t+h, x(t+h,\omega),\omega) - V(t,x(t,\omega),\omega) \\ &= V(t+h, \phi(0) + hf(t,\phi(0),\phi,\omega) \\ &\quad - V(t,\phi(0,\omega),\omega) \\ &\quad + V(t+h, x(t+h,\omega),\omega) \\ &\quad - V(t+h, \phi(0) + hf(t,\phi(0),\phi,\omega),\omega). \end{aligned}$$

This together with the hypotheses and the argument used in Theorem 4.1 in Ref. 15 yields the inequality

$$m'(t,\omega) \leq g(t, m(t,\omega), m_t(\omega),\omega) \quad (4.8)$$

almost everywhere in (t,ω). From the application of Theorem 3.1, we deduce that

$$V(t, x(t_0,\phi_0)(t,\omega),\omega) \leq r(t_0,\sigma_0)(t,\omega)$$

as long as $x(t,\omega) \in D(S(R^n))$ to the right of t_0. The proof is complete.

The following variant of Theorem 4.1 is often more useful in applications.

Theorem 4.2: Let the hypotheses of Theorem 4.1 hold except (H_4) is replaced by (H_5). Then, $V_{t_0}(\omega) \leq \sigma_0(\omega)$ implies

$$V(t, x(t_0,\phi_0)(t,\omega),\omega) \leq R(t_0,\psi_0)(t,\omega) \quad (4.9)$$

as long as $x(t,\omega) \in D(S(R^n))$, where $R(t_0,\psi_0)(t,\omega)$ is the maximal solution process of the auxiliary random functional differential system

$$\begin{aligned} v'(t,\omega) &= A^{-1}(t,\omega)[-A(t,\omega)v(t,\omega) \\ &\quad + g(t, A(t,\omega)v(t,\omega), (Av)_t(\omega),\omega)], \\ v_{t_0}(\omega) &= \psi_0(\omega) \end{aligned} \quad (4.10)$$

existing for $t \geq t_0$.

Proof: By following the argument used in Theorem 2 in Ref. 19 and Theorem 4.1, the proof of the theorem can be constructed, analogously.

Remark 4.1: Theorems 4.1 and 4.2 are analogous to Theorems 1 and 2[19] for deterministic functional differential systems, and are similar to Theorems 4.1 and 4.2[15] for nonhereditary random differential systems.

Now, we formulate a comparison theorem that is based on a random vector Lyapunov function and a minimal subset of $\mathcal{C}^n$.

Theorem 4.3: Assume that the hypotheses (H_6), (H_7), and (H_9) hold. Further assume that $x(t_0, \phi_0)(t, \omega) = x(t, \omega)$ be any solution process of (2.1) such that $\phi_0 \in S(C)$ and

$$\sup_{-\tau \le s \le 0} A(t_0 + s, \omega) V(t_0 + s, \phi_0(s, \omega), \omega) \le u_0(\omega). \quad (4.11)$$

Then,

$$A(t, \omega) V(t, x(t_0, \phi_0)(t, \omega), \omega) \le r(t, t_0, u_0, \omega), \quad (4.12)$$

as long as $x(t, \omega) \in D(S(R^n))$ w.p.1 for $t \ge t_0$.

Proof: Let $x(t, \omega) = x(t_0, \phi_0)(t, \omega)$ be any solution process of (2.1) satisfying (4.11) and $\phi_0 \in S(C)$. Define the random vector function

$$m(t, \omega) = A(t, \omega) V(t, x(t, \omega), \omega). \quad (4.13)$$

From (4.11) and (4.13), it is obvious that

$$|(A_i V) t_0(\omega)|_0 \le u_{i0}(\omega) \quad \text{for } i \in I, \quad (4.14)$$

where u_{i0} is the ith component of u_0.

For sufficiently small $\epsilon > 0$, consider the system of random differential equations

$$\dot{u}_i(t, \omega) = g_i(t, u(t, \omega), \omega) + \epsilon, \quad u_i(t_0, \omega) = u_{i0}(\omega) + \epsilon. \quad (4.15)$$

Let $u(t, \epsilon, \omega)$ be a solution of (4.15) existing as far as $r(t, \omega)$ exists to the right of t_0, where $r(t, \omega)$ is maximal solution of (2.3). Since

$$\lim_{\epsilon \to 0} u(t, \epsilon, \omega) = r(t, \omega) \quad \text{w.p.1},$$

the validity of the inequality

$$A(t, \omega) V(t, x(t, \omega), \omega) \le r(t, \omega) \quad (4.16)$$

is immediate as long as $x(t, \omega) \in D(S(R^n))$ and

$$m(t, \omega) \le u(t, \epsilon, \omega). \quad (4.17)$$

Our objective is to show that (4.17) is true. If we assume that (4.17) is false, then

$$z = \bigcup_{i=1}^{n} \{t \in R_+ : m_i(t, \omega) \ge u_i(t, \epsilon, \omega) \text{ w.p.1}\}$$

is nonempty. Let $t_1 = \inf z$. Arguing as in Theorem 1.5.1[21] there exists an index i, $t_1 > t_0$ and $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that

(i) $m_i(t_1, \omega) = u_i(t_1, \epsilon, \omega)$, $\omega \in \Omega_1$,

(ii) $m_i(t, \omega) < u_i(t, \epsilon, \omega)$, $t \in [t_0, t_1)$ and $\omega \in \Omega_1$,

(iii) $m_j(t, \omega) \le u_j(t, \epsilon, \omega)$ w.p.1, $t \in [t_0, t_1]$, $i \ne j$.

Therefore

$$D^+ m_i(t_1, \omega) \ge \dot{u}_i(t, \epsilon, \omega) = g_i(t_1, u(t_1, \epsilon, \omega), \omega) + \epsilon. \quad (4.18)$$

Since $g(t, u, \omega) \ge 0$ w.p.1, $u(t, \epsilon, \omega)$ is nondecreasing in t with probability one. Consequently, it follows from (i), (ii), and (iii) that

$$\sup_{-\tau \le s \le 0} m_i(t_1 + s, \omega) = m_i(t_1, \omega) = u_i(t, \epsilon, \omega), \quad (4.19)$$

and

$$\sup_{-\tau \le s \le 0} m_j(t_1 + s, \omega) \le u_j(t_1, \epsilon, \omega), \quad i \ne j. \quad (4.20)$$

Setting $\phi(\omega) = x_{t1}(t_0, \phi_0)(\omega)$ and noting that $\phi(0, \omega) = x(t_1, \omega)$, we have

$$|(A_i V)_{t_1}(\omega)|_0 = A_i(t_1, \omega) V(t_1, \phi(0, \omega), \omega). \quad (4.21)$$

This implies that $\phi(\omega) \in S(\Omega_{iA})$. Hence, using the Lipschitzian nature of $V(t, x, \omega)$ in x, the relation (4.4), the quasimonotone property of $g(t, u, \omega)$ in u, and the inequalities (4.19), (4.20), and (4.21), we have the following inequality:

$$D^+ m_i(t_1, \omega) \le g_i(t_1, u(t_1, \epsilon, \omega), \omega).$$

This inequality is incompatible with (4.18), and hence the set z is emply, which in turn proves the validity of inequality (4.17). This completes the proof of the theorem.

Remark 4.2: We observe that comparison Theorems 4.1 and 4.3 differ in the following sense. Theorem 4.1 gives the estimate for the solutions of (2.1) with respect to the maximal solution of its auxiliarly random functional differential system (2.2). On the other hand, Theorem 4.3 gives the estimate for the solutions of (2.1) with respect to the maximal solution of its auxiliary random ordinary differential system.

Remark 4.3: We note that estimate (4.12) is equivalent to

$$V(t, x(t_0, \phi_0)(t, \omega), \omega) \le R(t, t_0, v_0, \omega), \quad (4.22)$$

where $R(t, t_0, v_0, \omega)$ is the maximal solution of the auxiliary differential system

$$v'(t, \omega) = A^{-1}(t, \omega)[-A'(t, \omega) v(t, \omega) + g(t, A(t, \omega) \times v(t, \omega), \omega)] \quad v(t_0, \omega) = v_0(\omega) \quad (4.23)$$

existing for $t \ge t_0$, and moreover

$$A(t, \omega) R(t, t_0, v_0, \omega) = r(t, t_0, u_0, \omega) \quad (4.24)$$

with $u_0(\omega) = A(t_0, \omega) v_0(\omega)$.

5. STABILITY RESULTS

In this section, we employ the comparison theorems that are developed in the preceding section, to study stability properties of the trivial solution process of (2.1).

In the following, we present a few main results which are based on Theorem 4.1. The following result estab establishes the stability properties of (2.1) in the sense of probability and in the sense of probability one.

Theorem 5.1: Let the hypotheses (H_1), (H_2), (H_3), (H_4), and (H_{10}) be satisfied. Assume that $f(t, 0, 0, \omega) = 0$. Then,

(i) (SP_1^*) of (2.2) implies (SP_1),

(ii) (SP_1^*) of (2.2) implies (SP_2),

(iii) (SS_1^*) of (2.2) implies (SS_1),

(iv) (SS_2^*) of (2.2) implies (SS_2),

Proof: Let us prove the statement (i). Let $\eta > 0$, $0 < \epsilon < \rho$, and $t_0 \in R^+$ be given. Assume that (SP_1^*) of (2.2) holds. Then $b(\epsilon)$, $\eta > 0$ and $t_0 \in R^+$, there exists a positive number $\delta_1 = \delta_1(t_0, \epsilon, \eta)$ such that

$$P\{\omega : \sum_{i \in I} u_i(t_0, \sigma_0)(t, \omega) \ge b(\epsilon)\} < \eta, \quad t \ge t_0 \quad (5.1)$$

whenever

$$P\{\omega : \sup_{-\tau \le s \le 0} |\sum_{i \in I} \sigma_0(s,\omega)| > \delta_1\} < \eta. \tag{5.2}$$

We choose σ_0 such that $V(t_0+s, \phi_0(s,\omega),\omega) \le \sigma_0(s)$ and

$$\sum_{i \in I} \sigma_{0i}(s,\omega) = a(t_0+s, \|\phi_0(s,\omega)\|). \tag{5.3}$$

Since $a(t,\cdot) \in K$, for fixed $s \in [t_0, -\tau, t_0]$, we can find $\delta(t_0+s, \epsilon, \eta) = \delta_s > 0$ such that

$$P\{\omega : a(t_0+s, \|\phi_0(s,\omega)\|) > \delta_1\} = P\{\omega : \|\phi_0(s,\omega)\| > \delta_s\} < \eta. \tag{5.4}$$

Our aim is to choose δ which is independent of $s \in [-\tau, 0]$. From the sample continutity of $\|\phi_0(s,\omega)\|$ is s for each $s \in [-\tau, 0]$, we can find η_s such that

$$P\{\omega : \|\phi_0(\theta,\omega)\| > \delta_s\} < \eta \text{ for } \theta \in (-\eta_s + s, \eta_s + s) \cap [-\tau, 0]$$

This is true for each $s \in [-\tau, 0]$. Consider the collection of open sets in $[-\tau, 0]$ defined by

$$U = \{0_s : 0_s = (-\eta_s + s, \eta_s + s) \cap [-\tau, 0] \text{ for } s \in [-\tau, 0]\}.$$

It is easy to verify that it is an open covering of $[-\tau, 0]$ and hence by Heine—Borel theorem, we can extract a finite subcover corresponding to $s_1, s_2, \ldots, s_k$ for some fixed integer k. Take the corresponding numbers $\delta_{s_1}, \delta_{s_2}, \ldots, \delta_{s_k}$ and set

$$\delta = \min\{\delta_{s_1}, \delta_{s_2}, \ldots, \delta_{s_k}\}$$

Then, we have

$$P\{\omega : \sup_{-\tau \le s \le 0} a(t_0+s, \|\phi_0(s,\omega)\|, \omega) > \delta_1\}$$
$$= P\{\omega : \sup_{-\tau \le s \le 0} \|\phi_0(s,\omega)\| > \delta\} < \eta. \tag{5.4}$$

Now, we claim that (SP_1) holds. Suppose that this is flase. Then there would exist a solution process $x(t,\omega)$ of (2.1) with

$$P\{\omega : \sup_{-\tau \le s \le 0} \|\phi_0(s,\omega)\| > \delta\} < \eta \text{ and } t_1 > t_0$$

such that

$$P\{\omega : \|x(t_1,\omega)\| \ge \epsilon\} = \eta, \tag{5.5}$$

and $x(t,\omega) \in D(S(R^n))$ w.p.1 for $t \in [t_0, t_1]$. On the other hand, by Theorem 4.1, the inequality

$$V(t, x(t,\omega), \omega) \le r(t,\omega) \tag{5.6}$$

is valid as long as $x(t,\omega) \in D(S(R^n))$ w.p.1. From (4.5) and (5.6), we have

$$b(\|x(t,\omega)\|) \le \sum_{i \in I} V_i(t, x(t,\omega), \omega)$$
$$\le \sum_{i \in I} r_i(t,\omega). \tag{5.7}$$

The relations (5.1), (5.5), and (5.7) lead us to the contradiction

$$\eta \le P\{\omega : \sum_{i \in I} V_i(t_1, x(t_1,\omega), \omega) \ge b(\epsilon)\}$$
$$\le P\{\omega : \sum_{i \in I} r_i(t_1,\omega) \ge b(\epsilon)\} < \eta.$$

This proves the statement (i).

The proof of statement (ii) can be constructed by following the proof of statement (i) and the proof of Theorem 5.1 in Ref. 15. The proofs of statements (iii) and (v) can be formulated by following the argument used in proofs of statements (i) and (ii), and the deterministic version[19] of the theorems. To avoid monotone, we omit the details.

Corollary 5.1: Assume that the hypotheses of Theorem 5.1 hold except that (H_{10}) is replaced by

(H_{10}^*) for $(t,x) \in [-\tau, \infty] \times D$,

$$b(\|x\|) \le \sum_{i \in I} V_i(t,x,\omega) \le a(t, \|x\|, \omega),$$

where b, $a(t,\cdot,\omega) \in K$ and $a(t,r,\omega)$ is sample continuous random function in (t,r). Under this modification, the conclusions of Theorem 5.1 remain true.

Proof: The proof or the corollary follows by following the proof of Theorem 5.1 directly.

The following result establishes the stability properties of (2.1) in the sense of first moment.

Theorem 5.2: Assume that the hypotheses of Theorem 5.1 hold except that (H_{10}) is replaced by (H_{11}). Then

(i) (SM_1^*) of (2.2) implies (SM_1),

(ii) (SM_2^*) of (2.2) implies (SM_2).

Proof: Let us prove the statement (i). Let $\rho > \epsilon > 0$, $t_0 \in R_+$ be given. Assume that (SM_1^*) holds. Then $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \epsilon)$ such that

$$\sup_{-\tau \le s \le 0} \sum_{i \in I} E[\sigma_{0i}(s,\omega)] \le \delta_1 \tag{5.8}$$

implies

$$\sum_{i \in I} E[u_i(t_0, \sigma_0)(t,\omega)] < b(\epsilon), \quad t \ge t_0. \tag{5.9}$$

We choose σ_0 such that $V(t_0+s, \phi_0(s,\omega), \omega) \le \sigma_0(s,\omega)$ and

$$\sum_{i \in I} E[\sigma_{0i}(s,\omega)] = a(t_0+s, E[\|\phi_0(s,\omega)\|]). \tag{5.10}$$

Now by following the argument used in the proof of Theorem 5.1 or more precisely, the argument similar to the argument used in Theorem 4.1 in Ref. 8, one can find a positive number $\delta = \delta(t_0, \epsilon)$ such that

$$\sup_{-\tau \le s \le 0} E[\|\phi_0(s,\omega)\|] \le \delta$$

implies

$$a(t_0+s, E[\|\phi_0(s,\omega)\|] < \delta_1$$

for $s \in [-\tau, 0]$. Now the rest of the proof of the theorem can be completed by using the argument that is used in the proof of Theorem 5.2 in Ref. 15. Thus completing the proof of the statement (i).

The proof of statement (ii) can be constructed by using the arguments that are used in the proofs of Theorem 4.1 in Ref. 8 and Theorem 5.1 in Ref. 15.

In general, we may not be able to find the auxiliary random functional differential system (2.2) whose trivial solution has (SP*), (SM*), and (SS*) properties. In such case, the comparison Theorem 4.2 is useful in discussing the stability properties of (2.1). In the following, we state the results whose proofs can be constructed by following proofs of theorems in Refs. 8 and 15.

Theorem 5.3: Assume that the hypotheses of Theorem 5.1 hold except that (H_4) is replaced by (H_5). Then,

(i) (SP_1^*) of (4.10) implies (SP_1),

(ii) (SP_2^*) of (4.10) implies (SP_2),

(iii) (SS_1^*) of (4.10) implies (SS_1),

(iv) (SS_2^*) of (4.10) implies (SS_2).

Theorem 5.4: Assume that the hypotheses of Theorem 5.3 hold except that (H_{10}) is replaced by (H_{11}). Then,

(i) (SM_1^*) of (4.10) implies (SM_1),

(ii) (SM_2^*) of (4.10) implies (SM_2).

In the following, we present two main stability results which are based on Theorem 4.3.

Theorem 5.5: Let the hypotheses (H_6), (H_7), (H_8), (H_9), and (H_{10}) be satisfied. Further assume that f in (2.1) satisfies $f(t,0,0,\omega)=0$. Then,

(i) (SP_1^*) of (4.23) implies (SP_1),

(ii) (SP_2^*) of (4.23) implies (SP_2),

(iii) (SS_1^*) of (4.23) implies (SS_1),

(iv) (SS_2^*) of (4.23) implies (SS_2).

Proof: Let us prove statement (i). Let $\rho > \epsilon > 0$, $\eta > 0$, and $t_0 \in R_+$ be given. Assume that (SP_1^*) of (4.23) holds. Then $b(\epsilon) > 0$, $\eta > 0$, and $t_0 \in R_+$, there exists a positive number $\delta_1 = \delta_1(t_0, \epsilon, \eta)$ such that

$$P\{\omega: \sum_{i\in I} u_{0i}(\omega) > \delta_1\} < \eta \tag{5.11}$$

impoiles

$$P\{\omega: \sum_{i\in I} u_i(t,\omega) \geq b(\epsilon)\} < \eta, \quad t \geq t_0. \tag{5.12}$$

We choose $u_0(\omega)$ such that

$$\sup_{-\tau\leq s\leq 0} A(t_0+s,\omega)V(t_0+s,\phi_0(s,\omega),\omega) \leq u_0(\omega)$$

and

$$\sum_{i\in I} u_{0i}(\omega) = a(t_0, \sup_{-\tau\leq s\leq 0} \|\phi_0(s,\omega)\|). \tag{5.13}$$

Since $a(t_0,\cdot) \in \mathcal{K}$, we can find a positive number $\delta = \delta(t_0, \epsilon, \eta)$ such that

$$P\{\omega: a(t_0, \sup_{-\tau\leq s\leq 0} \|\phi_0(s,\omega)\|) > \delta_1\}$$
$$= P\{\omega: \sup_{-\tau\leq s\leq 0} \|\phi_0(s,\omega)\| > \delta\} < \eta. \tag{5.14}$$

Now the remaining proof of (SP_1) of (2.1) follows from the proof of statement (i) in Theorem 5.1. Therefore, we omit the details. The proofs of statements (ii)—(iv) can constructed by following the proof of statement (i) and the proofs of statements (ii)—(iv) in Theorem 5.1.

Theorem 5.6: Assume that the hypotheses of Theorem 5.5 hold except that (H_{10}) is replaced by (H_{11}). Then,

(i) (SM_1^*) of (4.23) implies (SM_1)

(ii) (SM_2^*) of (4.23) implies (SM_2).

Proof: The proof of the theorem can be constructed from the proofs of Theorems 5.2 and 5.5. To avoid monotony, we leave the details to the reader.

Remark 5.1: We observe that in Theorems 5.5 and 5.6 the $a(t,r)$'s in hypotheses (H_{10}) and (H_{11}), respectively, need not have to be defined on $[-\tau,\infty) \times R_+$, instead, it is enough to be defined on $R_+ \times R_+$. Furthermore, a corollary to Theorem 5.5, similar to Corollary 5.1 can be stated, analogously.

Remark 5.2: Note that one can formulate the results corresponding to uniform notions under the hypotheses of the previous results except that (H_{10}) and (H_{11}) are replaced by (H_{12}) and (H_{13}), respectively, and the corresponding notions relative to auxiliary equations (2.2), (4.10), and (4.23) are uniform.

Remark 5.3: Theorems 5.1 and 5.2 are extensions of Theorems 5.1, 5.2, and 5.3 in Ref. 15, and are natural extensions of deterministic results in Ref. 19. Theorems 5.5 and 5.6 are extensions and generalizations of deterministic results in Ref. 21.

Remark 5.4: We also note that our stability results are local in nature. If, $\rho = \infty$, then $D = R^n$, and the previous stability results would be of global character.

Remark 5.5: We further note that our preceding discussion includes the discussion of random functional differential systems of the type

$$x'(t,\omega) = F(t, x_t(\omega), \omega), \quad x_{t_0}(\omega) = \phi_0(\omega),$$

where $x_t(\omega) = x(t+s,\omega)$, $y(t) \leq s \leq 0$, $-\tau \leq y(t) \leq 0$, and $\{y(t): t \in R_+\}$ is a random process defined on (Ω, F, P) into $[-\tau, 0]$, that is, systems with time-varying random delays.

6. EXAMPLES

In this section, we shall present some examples that demonstrate the scope of our results.

Example 6.1: Consider the system of random functional differential equations

$$x'(t,\omega) = f(t,\omega)x(t,\omega) + F(t, x(t-\tau,\omega),\omega), \tag{6.1}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(t,\omega) = \begin{bmatrix} -f_1(t,\omega) & 0 \\ 0 & -f_2(t,\omega) \end{bmatrix},$$

$f_i \in M[R_+, S(R_+)]$ for $i = 1, 2$ and locally sample Lebesque integrable on R_+;

$$F(t, x(t-\tau,\omega),\omega) = \begin{bmatrix} F_1(t, x(t-\tau,\omega),\omega) \\ F_2(t, x(t-\tau,\omega),\omega) \end{bmatrix},$$

it satisfies

$$|F_i(t, x(t-\tau,\omega)| \leq \lambda(t,\omega) \sum_{i=1}^{2} |x_i(t-\tau),\omega| \text{ w.p.1.} \tag{6.2}$$

and $F(t,0,\omega) \equiv 0$ w.p.1, where $\lambda \in M[R_+, S(R_+)]$. Take $m = 2$ and

$$V(t,x,\omega) = \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}.$$

In view of the assumptions on (6.1), we have

$D^+V(t,\phi(0),\phi)$

$$\leq f(t,\omega)V(t,\phi(0),\omega)+\Lambda(t,\omega)V(t,\phi(-\tau),\omega). \quad (6.3)$$

where

$$\Lambda(t,\omega)=\begin{bmatrix}\lambda(t,\omega) & \lambda(t,\omega)\\ \lambda(t,\omega) & \lambda(t,\omega)\end{bmatrix}.$$

The auxiliary system is $u'(t,\omega)=f(t,\omega)u(t,\omega)+\Lambda(t,\omega)u(t-\tau,\omega)$. The trivial solution process $u\equiv 0$ is

(i) stable in probability, if

$$\lim_{t\to\infty}P\{\omega:[\int_{t_0}^{t}(2\lambda(s,\omega)\exp[\int_{s-\tau}^{s}f_*(u,\omega)du] - f_*(s,\omega)ds]\geq k\}=0, \quad (6.4)$$

where $k=k(t_0)$ is some positive real number, and $f_*=\min\{f_1,f_2\}$;

(ii) asymptotically stable in probability, if

$$\lim_{t\to\infty}P\{\omega:[\int_{t_0}^{t}(2\lambda(s,\omega)\exp[\int_{s-\tau}^{s}f_*(u,u)du] - f_*(s,\omega))ds]\geq 0\}=0; \quad (6.5)$$

(iii) stable with probability one, if

$$P\{\omega:\lim_{t\to\infty}[\int_{t_0}^{t}(2\lambda(s,\omega)\exp[\int_{s-\tau}^{s}f_*(u,\omega)du] - f_*(s,\omega))ds]\leq k\}=1, \quad (6.6)$$

where $k=k(t_0)$ is some positive real number;

(iv) asymptotically stable with probability one, if

$$P\{\omega:\lim_{t\to\infty}\left[\frac{1}{t-t_0}\left(\int_{t_0}^{t}(2\lambda(s,\omega)\exp[\int_{s-\tau}^{s}f_*(u,\omega)du\right) - f_*(s,\omega)ds)\right]\leq -a\}=1, \quad (6.7)$$

where a is some positive real number, and $f_*=\min\{f_1,f_2\}$;

Hence by Theorem 5.1, it follows that (6.4) implies (SP_1) of (6.1), (6.5) implies (SP_2) of (6.1), (6.6) implies (SS_1) of (6.1), and (6.7) implies (SS_2) of (6.1).

The following example illustrates the use of Comparision Theorem 4.3.

Example 6.2: Consider the system random functional differential equations

$$x'(t,\omega)=L(t,x_t(\omega),\omega) \quad (6.8)$$

where $x\in R^n$, $L\in C[R_+\times C^n, S(R^n)]$ and $L(t,\phi,\omega)$ is almost surely linear functional in ϕ. By Riesz Theorem,[22] we have

$$x'(t,\omega)=\int_{-\tau}^{0}d_s\eta(t,\omega,s)(t+s,\omega), \quad (6.9)$$

where $\eta(t,\omega,s)$ is an $n\times n$ random matrix whose elements $\eta_{ij}=\eta_{ij}(t,\omega,s)$ are almost surely sample continuous random functions in $t\in R_+$ and are almost surely sample functions of bounded variation on $[-\tau,0]$; the integral is almost surely sample Stieltjes integral on $[-\tau,0]$. For $t\geq t_0+\tau$, (6.9) can be written as

$$x'(t,\omega)=\int_{-\tau}^{0}d_s\eta(t,s)(t,\omega)-\int_{-\tau}^{0}d_s\eta(t,s)[\int_{t+s}^{t}[\int_{-\tau}^{0}d_\theta\eta(u,\theta)(u+\theta)]du]. \quad (6.10)$$

Take $V(t,x,\omega)=\sum_{i=I}^{n}|x_i|$. From (6.8), (6.9), and (6.10), for $h>0$ and $t\geq t_0+t$, we have

$$V(t+h,x+hL(t,x_t,\omega),\omega)$$
$$=\sum_{i=I}^{n}|x_i+h\sum_{k=1}^{n}\int_{-\tau}^{0}d_s\eta_{ik}(t,\omega,s)x_k(t+s)|$$
$$=\sum_{i=1}^{n}[|x_i+h\sum_{k=1}^{n}\int_{-\tau}^{0}d_s\eta_{ik}(t,\omega,s)\times[x_k-\sum_{j=1}^{n}\int_{t+s}^{t}[\int_{-\tau}^{0}d_\theta\eta_{kj}(u,\omega,\theta)x_j(u+\theta)]du]|]$$
$$\leq\sum_{i=1}^{n}[|x_i+h\sum_{k=1}^{n}\int_{-\tau}^{0}d_s\eta_{ik}x_k|+h|\sum_{k=1}^{n}[\sum_{j=1}^{n}\int_{-\tau}^{0}d_s\eta_{ik}\times[\int_{t+s}^{t}[\int_{-\tau}^{0}d_\theta\eta_{kj}x_j(u+\theta)]du]]|]. \quad (6.11)$$

The first and second terms in the right-hand side of the inequality (6.11) can be written as

$$|1+h\int_{-\tau}^{0}d_s\eta_{ii}|\ |x_i|+h\sum_{\substack{k=1\\k\neq i}}^{n}|\int_{-\tau}^{0}d_s\eta_{ik}|\ |x_k| \quad (6.12)$$

and

$$h\sum_{k=1}^{n}[\sum_{j=1}^{n}\tau\int_{-\tau}^{0}|d_s\eta_{ik}|\sup_{t-\tau\leq s\leq t}\int_{-\tau}^{0}|d_\theta\eta_{kj}|)\times(\sup_{-2\tau\leq s\leq 0}|x_j(t+s)|)], \quad (6.13)$$

respectively. From (11), (12), and (13), one obtains

$$D^+V(t,\phi(0),\phi,\omega)\leq\sup_k[a_{kk}(t,\tau,\omega)+\sum_{\substack{i=1\\i\neq k}}^{n}a_{ik}(t,\tau,\omega)]V(t,\phi(0),\omega)+\tau\sum_{i=1}^{n}\sum_{k=1}^{n}b_{ik}(t,\tau,\omega)\sup_{-2\tau\leq s\leq 0}V(t+s,\phi(s),\omega) \quad (6.14)$$

where $a_{ij}(t,\tau,\omega)$, $b_{ij}(t,\tau,\omega)$ are defined by

$$a_{ij}(t,\tau,\omega)=\begin{cases}\int_{-\tau}^{0}d_s\eta_{jj}(t,\omega,s), & i=j,\\ |\int_{-\tau}^{0}d_s\eta_{ij}(t,\omega,s)|, & i\neq j;\end{cases}$$

$b_{ij}(t,\tau)$ is the entry in the ith row and jth column of the $n\times n$ matrix $T(\eta(t,\omega,s))\cdot S(T(\eta(t+u,\omega,s))$, $T(\eta(t,\omega,s))=(T(\eta_{ij}(t,\omega,s)))$ and $S(T(\eta(t+u,\omega,s))=(\sup_{-\tau\leq u\leq 0}T(\eta_{ij}(t+u,\omega,s)))$, $T(\eta_{ij}(t,\omega,s))$ is the total variation of the ith and jth entry $\eta_{ij}(t,\omega)$ of the matrix $\eta(t,\omega)$ on the interval $[-\tau,0]$.

We assume that

$$a_{ij}(t,\tau,\omega)<0 \text{ w.p.1}, \quad (6.15)$$

and

$$|a_{ij}(t,\tau,\omega)|-\sum_{\substack{i=1\\i\neq j}}^{n}a_{ij}(t,\tau,\omega)-\tau[\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ij}(t,\tau,\omega)]\geq k \quad (6.16)$$

for some $j=1,2,\ldots,n$ and some positive real number k.

For any $\alpha\in[0,\infty)$, then the function

$$F(\alpha)=\alpha+\sup_j[a_{jj}(t,\tau,\omega)+\sum_{\substack{i=1\\i\neq j}}^{n}a_{ij}(t,\tau,\omega)]+\tau\exp(2\alpha\tau)[\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ij}(t,\tau,\omega)]$$

defined and continuous on $[0,\infty)$. From (6.15) and (6.16), and the definition of F, we have $F(0)<0$. Moreover, $F(\alpha)$ is increasing on $[0,\infty)$, therefore we can find a positive number λ such that

$$F(\lambda) \leq 0. \tag{6.17}$$

We take $A(t,\omega)\exp[\lambda t]$, and define

$$\Omega_A = \{\phi \in C^n: \sup_{-\tau \leq s \leq 0} A(t+s,\omega)V(t+s,\phi(s),\omega) = A(t,\omega)V(\phi(0))\}$$

From (6.14), (6.17), the definition of V, and

$$\sup_{-2\tau \leq s \leq 0} |\phi_i(s)| \leq \sup_{-2\tau \leq s \leq 0} \|\phi(s)\| \leq \|\phi(0)\| \exp(2\lambda t),$$

we have

$$A(t,\omega)D^+V(t,\phi(0),\phi,\omega) + A'(t,\omega)V(t,\phi(0),\omega) \leq 0 \tag{6.18}$$

whenever $\phi \in \Omega_A$. The comparison equation (4.23) reduces to $u'(t,\omega) = -\lambda u(t,\omega)$. It is obvious that the trivial solution of this is almost surely sample asymptotically stable. Therefore, by the application of Theorem 5.5, the trivial solution of (6.8) is almost surely sample asymptotically stable.

To illustrate the usefulness of comparison theorems relative to the system (2.1), one can construct examples similar to Example 6.3 in Ref. 13. In addition, to show an advantage of a vector Lyapunov function, an example similar to Example 5.3 in Ref. 17 can be given, analogously. To avoid repetition, we do not want to present further details.

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The Feynman maps and the Wiener integral

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By introducing the family of Feynman maps $\mathcal{J}^s$, we show that our earlier definition of the Feynman path integral $\mathcal{J} = \mathcal{J}^1$ can be obtained as the analytic continuation of the Wiener integral $E = \mathcal{J}^{-i}$. This leads to some new results for the Wiener and Feynman integrals. We establish a translation and Cameron–Martin formula for the Feynman maps $\mathcal{J}^s$, having applications to nonrelativistic quantum mechanics. We also estalish a (weak) dominated convergence theorem for $\mathcal{J}^1 = \mathcal{J}$.

1. INTRODUCTION

In previous papers we have given a new definition of the Feynman path integral $\mathcal{J}$ in non-relativistic quantum mechanics. Unlike most existing definitions of the path integral our formulation is based on the piecewise linear polygonal paths introduced to the subject by Feynman.[1] Apart from its physical appeal, the simplicity of our definition makes it very easy in principle to evaluate Feynman integrals. In spite of the naivety of this definition it has some nontrival applications to quantum theory.[2]

We have seen for instance that the wavefunction solution of the Cauchy problem for the Schrödinger equation, for harmonic oscillator potentials $V(x) = Ax^2 + Bx + C$, $A \geq 0$, and continuous bounded potentials $V(x) = \int \exp(i\alpha x)\, d\mu(\alpha)$, where μ is a measure of bounded absolute variation, can be expressed as a Feynman integral.[3] For these potentials this validates the Feynman—Dirac conjecture expressing the quantum mechanical amplitude as a "sum over paths γ" of $\exp\{iS[\gamma]/\hbar\}$, where $S[\gamma]$ is the classical action corresponding to the path γ and $\hbar$ is Planck's constant divided by 2π.

We have also seen that, using the quasiclassical representation which we introduced in an earlier publication, the Feynman path integral $\mathcal{J}$ gives a simple way of obtaining classical mechanics as the limiting case of quantum mechanics when $\hbar \to 0$.[3] Hence $\mathcal{J}$ fulfils one of the early hopes of Feynman and Dirac that the path integral approach to quantum mechanics should give the classical mechanical limit in a straightforward way.

To extend the applicability of the Feynman path integral $\mathcal{J}$, it is necessary now to establish a body of theorems to simplify the evaluation of Feynman integrals, or to reduce their evaluation to more or less standard mathematical procedures. This task is undertaken in the present paper.

We define here a family of Feynman maps $\mathcal{J}^s$ and show that the Feynman path integral $\mathcal{J} = \mathcal{J}^1$ can be obtained as the analytic continuation of the Wiener integral $E = \mathcal{J}^{-i}$. This leads to new results for both the Feynman and Wiener integrals, summarized in Theorems 3 and 4. The main content of the present paper is the proof of a Cameron—Martin formula for the Feynman maps $\mathcal{J}^s$ and a (weak) dominated convergence theorem for $\mathcal{J}$. These are given in Theorems 6 and 7. Both these results extend the applicability of $\mathcal{J}$. We shall discuss their applications to nonrelativistic quantum mechanics in a future paper. Paralleling the development of the Wiener integral, we also establish a translation formula for the Feynman maps $\mathcal{J}^s$ in Theorem 5. Because of the simplicity of our definitions, all these results are fairly easy to establish. We feel that this is one of the strong points in favor of our treatment.

In writing this paper we have been strongly influenced by the papers of Nelson[4] and Albeverio and Høegh-Krohn.[5] It seems appropriate at this stage to list one or two other references. For early work on the rigorous definition of the Feynman integral we refer the reader to Cameron's papers whose approach is similar in spirit to ours, but whose results demand analytic potentials.[6] The aforementioned paper of Nelson gives a rigorous definition of a Feynman path integral with extensive applications to the Schrödinger equation, but it does not define the Feynman integral of general functionals on path space.

More recently Itô has defined a path integral as a limit of certain Gaussian measures on a Hilbert path space.[7] Tarski has written on this formulation and given a number of applications.[8] Although there are close connections with our work, we feel that Itô's approach is less intuitive and more difficult to handle than the one which we advocate.

The above work of Albeverio and Høegh-Krohn defines a very elegant path integral by means of infinite-dimensional oscillatory integrals.[9] This definition makes great use of the Fourier transform on path space, the idea for which also appeared in the earlier distribution-theoretic work of DeWitt-Morette.[10] In some ways our work can be regarded as a synthesis of the ideas of Cameron and Albeverio and Høegh-Krohn. This synthesis is possible because of the simple connection between the reproducing kernel of the underlying Hilbert path space and the piecewise linear polygonal paths. The reproducing kernel is an important ingredient in simplifying most of our proofs. For a review of work on the Feynman path integral until 1974 we cite Tarski and for an earlier review Gel'fand and Yaglom.[11]

Finally we add that, to make our exposition as simple as possible, we have restricted our attention to one-dimensional path integrals. It is a simple matter to generalize our results to n-dimensional flat-space path integrals.[12] To make the paper as self-contained

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as possible, we have included one or two of our earlier results—most notably Theorems 1 and 2.

2. THE FEYNMAN MAPS

In this section we establish what we feel is an important connection between the polygonal path formulation of Feynman integrals[6] and the more recent path-space Fourier transform approach to the subject.[5,10] This connection is crucial for our subsequent results.

The Hilbert space of paths H will be fundamental in all that follows. H is the space of continuous functions $\gamma:(0,t)\to\mathbb{R}$ with weak derivative $d\gamma/d\tau\in L^2(0,t)$, normalized so that $\gamma(t)=0$, endowed with inner product $(\ ,\ )$

$$(\gamma,\gamma')=\int_0^t \frac{d\gamma}{d\tau}\frac{d\gamma'}{d\tau}\,d\tau. \tag{1}$$

We recapitulate the main properties of H in Theorem 1.

Theorem 1: H is a real separable Hilbert space in inner product norm topology. $\gamma\in H$ iff $\exists$ constants $\alpha_0,\alpha_n,\beta_n\in\mathbb{R}$ with $\sum_1^\infty(\alpha_n^2+\beta_n^2)<\infty$ such that

$$\gamma(\tau)=\alpha_0(\tau-t)+\sum_{n=1}^\infty \frac{\alpha_n t}{2\pi n}\sin\left(\frac{2\pi n\tau}{t}\right)+\sum_{n=1}^\infty\frac{\beta_n t}{2\pi n}\left[1-\cos\left(\frac{2\pi n\tau}{t}\right)\right],\quad \tau\in(0,t), \tag{2}$$

and

$$\|\gamma\|^2=(\gamma,\gamma)=t\alpha_0^2+\frac{t}{2}\sum_{n=1}^\infty(\alpha_n^2+\beta_n^2)<\infty. \tag{3}$$

H has a reproducing kernel $G(\sigma,\tau)=t-\sigma\check{}\tau$, where $\sigma\check{}\tau=\sup\{\sigma,\tau\}$, the reproducing property being, $\forall\ \gamma\in H$, $\forall\ \sigma\in[0,t]$,

$$(G(\sigma,\cdot),\gamma(\cdot))=\gamma(\sigma). \tag{4}$$

Proof: See Theorem 1 of Ref. 2(b).

The paths $\gamma\in H$ can be thought of as the paths which a quantum mechanical particle, in one space dimension, might actually describe in an experiment, but this point of view is far from mandatory.

We also require the linear maps $P_n:H\to H$, defined for $n=1,2,\cdots$, by

$$(P_n\gamma)(\tau)=\sum_{j=0}^{n-1}\left[G\left(\frac{\overline{j+1}t}{n},\tau\right)-G\left(\frac{jt}{n},\tau\right)\right]\times[\gamma_{j+1}-\gamma_j]\frac{n}{t}, \tag{5}$$

where $\gamma_j=\gamma(jt/n)$, $j=0,1,\ldots,n$. Evidently we have

$$(\gamma',P_n\gamma)=\sum_{j=0}^{n-1}(\gamma'_{j+1}-\gamma'_j)(\gamma_{j+1}-\gamma_j)\frac{n}{t}=(P_n\gamma',\gamma), \tag{6}$$

where $\gamma'_j=\gamma'(jt/n)$, $j=0,1,\ldots,n$. Substitution of $G(\sigma,\tau)=t-\sigma\check{}\tau$ gives

$$(P_n\gamma)(\tau)=\gamma_j+\left(\tau-\frac{jt}{n}\right)(\gamma_{j+1}-\gamma_j)\frac{n}{t},\qquad \frac{jt}{n}\le\tau<\frac{(j+1)t}{n}, \tag{7}$$

$j=0,1,2,\ldots,n-1$. It follows that $P_n^2=P_n$ and $P_n^*=P_n$. Hence $P_n:H\to H$ is a projection.

The paths $P_n\gamma$ are just the usual piecewise linear polygonal paths. The next theorem shows how numerous they are in H.

Theorem 2: $P_n:H$ is a projection. If $I:H\to H$ denotes the identity, then P_n tends to I in the strong operator topology on $\mathcal{L}(H,H)$, $P_n\xrightarrow{s}I$.

Proof: We must show that if $V=\{\gamma\in H\mid \|P_n\gamma-\gamma\|\to 0$ as $n\to\infty\}$, then $V=H$. We include here the proof which we first gave in Ref. 2(a). First V is a closed subspace of H. V is a subspace because P_n is linear. Let $\{\gamma_m\mid m=1,2,\cdots\}\subset V$ with $\|\gamma_m-\gamma\|\to 0$, as $m\to\infty$, for some $\gamma\in H$. We show that necessarily $\gamma\in V$. Observe that

$$\|P_n\gamma-\gamma\|=\|P_n\gamma-\gamma-P_n\gamma_m+\gamma_m+P_n\gamma_m-\gamma_m\|\le\|P_n(\gamma-\gamma_m)\|+\|\gamma-\gamma_m\|+\|P_n\gamma_m-\gamma_m\|. \tag{8}$$

Hence,

$$\|P_n\gamma-\gamma\|\le 2\|\gamma-\gamma_m\|+\|P_n\gamma_m-\gamma_m\|. \tag{9}$$

Given $\epsilon>0$, $\exists\ N_\epsilon$ such that $\|\gamma-\gamma_m\|<\epsilon/4$ when $m=N_\epsilon$. Also, $\exists\ N(m,\epsilon)$ such that $\|P_n\gamma_m-\gamma_m\|<\epsilon/2$, $n\ge N(m,\epsilon)$. From the above inequality then, for $n\ge N(N_\epsilon,\epsilon)$, $\|P_n\gamma-\gamma\|<\epsilon$. Thus, $\|P_n\gamma-\gamma\|\to 0$ as $n\to\infty$, so V is a closed subspace of H.

Now let $\gamma\in H$. Then $\exists\ \alpha_0,\alpha_n,\beta_n\in\mathbb{R}$ such that

$$\|d\gamma/d\tau-S_N\|_{L_2}\to 0,\quad \text{as } N\to\infty, \tag{10}$$

where

$$S_N(\tau)=\alpha_0+\sum_1^N\alpha_n\cos\left(\frac{2\pi n\tau}{t}\right)+\sum_1^N\beta_n\sin\left(\frac{2\pi n\tau}{t}\right). \tag{11}$$

Hence, integrating the above Fourier series term by term and defining $T_N(\tau)=-\int_\tau^t S_N(\tau')\,d\tau'$, we have

$$\|\gamma-T_N\|=\|d\gamma/d\tau-S_N\|_{L_2}\to 0,\quad \text{as } N\to\infty. \tag{12}$$

It is not difficult to show that, for each N, $T_N\in V$. Thus, $\gamma\in V$.

We now introduce the complex Gaussian $e_s:H\to\mathbb{C}$ defined by

$$e_s[\gamma]=\exp[(i/2s)\|\gamma\|^2],\quad s\in\mathbb{C},\qquad \text{with } \operatorname{Im}s\le 0,\ s\ne 0. \tag{13}$$

Let $f:H\to\mathbb{C}$ be a complex-valued functional. The Feynman map $\mathcal{F}^s$ is defined below.

Definition: For $n=1,2,\cdots$, we denote by $\mathcal{F}_n^s$

$$\mathcal{F}_n^s[f]=\int_{P_nH}(fe_s)\circ P_n d^n\gamma\left[\int_{P_nH}(e_s\circ P_n)d^n\gamma\right]^{-1}, \tag{14}$$

where $\circ$ denotes composition, $d^n\gamma=d\gamma_0\,d\gamma_1\cdots d\gamma_{n-1}$, each integration being from $-\infty$ to $+\infty$, and $\operatorname{Im}s\le 0$. (The normalization is chosen so that, for the functional 1, $\mathcal{F}_n^s[1]=1$.)

We define $\mathcal{F}^s[f]$, when $\operatorname{Im}s\le 0$, by

$$\mathcal{F}^s[f]=\lim_{n\to\infty}\mathcal{F}_n^s[f],\quad s\ne 0, \tag{15}$$

whenever the limit exists. We say that $f\in\mathcal{F}^s(P_\infty H)$ iff the above limit exists.

When $s=1$, the above definition reduces to that for $\mathcal{J}$ given in Ref. 2(b), where its physical content can be elucidated. Naturally it is important to ascertain for just how wide a class of functionals the above limit exists.

The first result in this direction is given in Theorem 3. First we require another definition.[13]

Definition: $\mathcal{J}(H)$ is the space of functionals $f: H\to\mathbb{C}$ with $f[\gamma]=\int_H \exp[i(\gamma',\gamma)]\,d\mu_f(\gamma')$, where $\mu_f\in M(H)$, the space of complex-valued measures of bounded absolute variation on $(H,\mathcal{B}_\sigma)$, $\mathcal{B}_\sigma$ being the Borel σ-field on H generated by open subsets of H.

Theorem 3: When $\mathrm{Im}\,s\le 0$, $\mathcal{J}(H)\subset\mathcal{J}^s(P_\infty H)$ and if $f\in\mathcal{J}(H)$ is given by

$$f[\gamma]=\int \exp[i(\gamma',\gamma)]\,d\mu_f(\gamma'),\quad \mu_f\in M(H),$$

$$\mathcal{J}^s[f]=\int \exp[-(is/2)\|\gamma\|^2]\,d\mu_f(\gamma). \tag{16}$$

Hence, for $\mathrm{Im}\,s\le 0$, we have

$$|\mathcal{J}^s[f]|\le\int_H d|\mu_f|\overset{\mathrm{def}}{=}\|f\|_0. \tag{17}$$

$\|\ \|_0$ is a norm on the Banach function algebra $\mathcal{J}(H)$ with identity 1. $\mathcal{J}^s:\mathcal{J}(H)\to\mathbb{C}$ is a continuous linear map, $\mathrm{Im}\,s\le 0$, normalized so that $\mathcal{J}^s[1]=1$.

Proof: When $\mathrm{Im}\,s<0$, the proof of the first part of the theorem is straightforward. Expressing P_n in terms of the reproducing kernel gives

$$[f\circ P_n][\gamma]=\int \exp\left(i\sum_{j=0}^{n-1}\Delta\gamma_j'\Delta\gamma_j\,\Delta t^{-1}\right)d\mu_f(\gamma'), \tag{18}$$

where $\Delta\gamma_j=(\gamma_{j+1}-\gamma_j)$, $\Delta\gamma_j'=(\gamma_{j+1}'-\gamma_j')$, $j=0,1,2,\ldots,$ $n-1$ and $\Delta t=t/n$.

Therefore, we obtain, for $\mathrm{Im}\,s\le 0$, $s\ne 0$,

$$\mathcal{J}_n^s[f]=(2\pi is\Delta t)^{-n/2}\int d^n\gamma\exp\left(\frac{i}{2s}\sum_{j=0}^{n-1}\Delta\gamma_j^2\,\Delta t^{-1}\right)\times\int\exp\left(i\sum_{j=0}^{n-1}\Delta\gamma_j'\Delta\gamma_j\Delta t^{-1}\right)d\mu_f(\gamma'). \tag{19}$$

Interchanging orders of integration by Fubini's theorem, for $\mathrm{Im}\,s<0$,

$$\mathcal{J}_n^s[f]=\int d\mu_f(\gamma')(2\pi is\Delta t)^{-n/2}\int d^n\gamma\times\exp\left(\frac{i}{2s\Delta t}\sum_{j=0}^{n-1}(\Delta\gamma_j^2+2s\Delta\gamma_j'\,\Delta\gamma_j)\right). \tag{20}$$

Completing the square in the exponential and evaluating the integral by contour integration, we obtain

$$\mathcal{J}_n^s[f]=\int d\mu_f(\gamma')\exp\left(\frac{-is}{2\Delta t}\sum_{j=0}^{n-1}\Delta\gamma_j'^2\right)=\int d\mu_f(\gamma')\exp\left(\frac{-is}{2}(\gamma',P_n\gamma')\right), \tag{21}$$

$\mathrm{Im}\,s<0.$

The result now follows from the last theorem and the dominated convergence theorem for μ_f.

When $\mathrm{Im}\,s=0$, $s\ne 0$, changing the orders of integration is slightly more delicate.

The argument which enables the orders of integration to be reversed was first given in Ref. 2(b). There, however, we did not give all the details of the proof that

$$f_R[\gamma']=N_n\exp\left(\frac{-i}{2}\sum_{j=0}^{n-1}\Delta\gamma_j'^2\,\Delta t^{-1}\right)\times\int_{-(R+\gamma_0')}^{R-\gamma_0'}\cdots\int_{-(R+\gamma_{n-1}')}^{R-\gamma_{n-1}'}\exp\left(\frac{i}{2}\sum_{j=0}^{n-1}\Delta\gamma_j''^2\,\Delta t^{-1}\right)d^n\gamma''$$

satisfies $|f_R[\gamma']|\le M$, independent of R and γ', the reason being that the details are somewhat tedious, depending upon repeatedly rotating and reflecting the region of integration. Here we avoid this technical detail altogether by a more careful definition of $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d^n\gamma$. We take $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d^n\gamma=\lim_{R\to\infty}\int_{-R}^{R}\cdots\int_{-R}^{R}d^n\,\Delta\gamma$, where the limits of integration $-R$ and $+R$ now refer to the integration variables $\Delta\gamma_0,\Delta\gamma_1,\ldots,\Delta\gamma_{n-1}$ not $\gamma_0,\gamma_1,\ldots,\gamma_{n-1}$, as previously. These are the natural integration variables in our later results too.

Fubini's theorem implies that, for real s, $s\ne 0$,

$$(2\pi is\,\Delta t)^{-n/2}\int_{-R}^{R}\cdots\int_{-R}^{R}\exp\left(\frac{i}{2s}\sum_{j=0}^{n-1}\Delta\gamma_j^2\,\Delta t^{-1}\right)\times[f\circ P_n][\gamma]d^n\,\Delta\gamma=\int d\mu_f(\gamma')f_R[\gamma'], \tag{19$'$}$$

where $f_R[\gamma']$ is now given by

$$f_R[\gamma']=(2\pi is\,\Delta t)^{-n/2}\exp\left(\frac{-is}{2}\sum_{j=0}^{n-1}\Delta\gamma_j'^2\,\Delta t^{-1}\right)\times\int_{-R+s\Delta\gamma_0'}^{R+s\Delta\gamma_0'}\cdots\int_{-R+s\Delta\gamma_{n-1}'}^{R+s\Delta\gamma_{n-1}'}\times\exp\left(\frac{i}{2s}\sum_{j=0}^{n-1}\Delta\gamma_j''^2\,\Delta t^{-1}\right)d^n\,\Delta\gamma'', \tag{20$'$}$$

γ_j'' being defined by $\gamma_j''=\gamma_j+s\gamma_j'$, $j=0,1,2,\ldots,n-1$ and $d^n\,\Delta\gamma''=d\Delta\gamma_0''\cdots d\Delta\gamma_{n-1}''$.

However, in Lemma 1 of Ref. 2(b), we establish that, for $b>0$, $|\int_0^a\exp(ibt^2)\,dt|\le C(b)$, uniformly $a\in(0,\infty)$. Hence, expressing the integral as a product, it follows that $|f_R[\gamma']|<M$, where M is independent of R and γ'. What is more, we easily see that $f_R[\gamma']\to\exp[-(is/2)\times(\gamma',P_n\gamma')]$ as $R\to\infty$. Applying the dominated convergence theorem for the measure $\mu_f\in M(H)$ in Eq. (19$'$) then yields, for real s, $s\ne 0$

$$\mathcal{J}_n^s[f]=\lim_{R\to\infty}(2\pi is\Delta t)^{-n/2}\int_{-R}^{R}\cdots\int_{-R}^{R}\times\exp\left(\frac{i}{2s}\sum_{j=0}^{n-1}\Delta\gamma_j^2\,\Delta t^{-1}\right)[f\circ P_n][\gamma]\,d^n\,\Delta\gamma=\int\exp\left(\frac{-is}{2}(\gamma',P_n\gamma')\right)d\mu_f(\gamma'), \tag{21$'$}$$

as required to change the orders of integration. The result follows letting $n\to\infty$ as above.

We now observe that $M(H)$ is a commutative Banach algebra in absolute variation $\|\ \|$, under the convolution of measures $*$. For by Fubini's theorem, for any bounded continuous functional f,

$$\int f[\gamma]d(\mu*\nu)(\gamma)=\int f[\gamma+\gamma']\,d\mu(\gamma)\,d\nu(\gamma'). \tag{22}$$

Hence we obtain

$$\mu * \nu = \nu * \mu, \quad \|\mu * \nu\| \le \|\mu\| \|\nu\|. \tag{23}$$

The associativity of $*$ follows from Fubini's theorem. Completeness follows by standard arguments.

We must now show that $\mathcal{J}(H)$ is the isometric isomorphic image of $M(H)$. The only difficult part of this result follows by putting $f[\gamma] = \exp[i(\gamma, \delta)]$ in above to obtain

$$\int \exp[i(\gamma, \delta)]\, d(\mu * \nu)(\gamma)$$

$$= \int \exp[i(\gamma, \delta)]\, d\mu(\gamma) \int \exp[i(\gamma', \delta)]\, d\nu(\gamma'). \tag{24}$$

If the entire function $E(z) = \sum_{n=0}^{\infty} a_n z^n$, then $E(f) = \sum_{n=0}^{\infty} a_n f^n$ is the Fourier transform of $\sum_{n=0}^{\infty} a_n (\mu_f * \mu_f * \cdots * \mu_f) \in M(H)$, as can be seen by repeated application of Eqs. (23) and (24). The separability of H implies that $\mu_f \in M(H)$ (if it exists) is uniquely determined by f and by definition $\|f\|_0 = \|\mu_f\|$. Thus, under multiplication and addition $\mathcal{J}(H)$ equipped with $\| \; \|_0$ is the isometric isomorphic image of the Banach algebra $M(H)$ and $\mathcal{J}(H)$ is a Banach algebra.

The continuity of $\mathcal{J}^s$ follows trivially from above. Finally $1 \in \mathcal{J}(H)$ is the Fourier transform of $\delta_0 \in M(H)$, where for Borel $A \subset H$,

$$\delta_0(A) = 1, \quad \text{if } 0 \in A, \qquad = 0, \quad \text{otherwise.} \tag{25}$$

Needless to say, the first term of $\sum_{n=0}^{\infty} a_n (\mu_f * \mu_f * \cdots * \mu_f)$ is understood to be $(a_0 \delta_0)$. δ_0 is just the identity in the Banach algebra $M(H)$ and 1 is the corresponding identity in the Banach algebra $\mathcal{J}(H)$. This completes the proof.

Corollary 1: $\forall f \in \mathcal{J}(H)$, $\lim_{s \to 0} \mathcal{J}^s[f] = f[0]$, where $s \to 0$ in the region $\mathrm{Im}\, s \le 0$.

Proof: Apply the dominated convergence theorem in Eq. (16). Henceforth, for any functional $f: H \to \mathbb{C}$ we define $\mathcal{J}^0[f] = f[0]$.[14]

The relationship in Theorem 3 is the promised connection between the "sum over polygonal paths" definition of the Feynman integral and the definition via "Fourier transform on path space" (see Ref. 2(b)].

3. THE WIENER INTEGRAL

In this section we explain the connection between the Feynman maps and the Wiener integral. We prove, in fact, that the Feynman path integral $\mathcal{J} = \mathcal{J}^1$ can be realized as an analytic continuation of the Wiener integral. This leads to new results for Wiener integrals as well as for Feynman path integrals.

First there is no *a priori* reason to restrict the path space to be H. Indeed since any $\gamma \in H$ is necessarily the integral of $d\gamma/d\tau \in L^2(0, t)$ and hence is absolutely continuous with a.e. derivative $d\gamma/d\tau$, the paths $\gamma \in H$ may well be too smooth to be consistent with Heisenberg's uncertainty principle. In the foregoing definition of the Feynman maps $\mathcal{J}^s$ we merely require that Γ, the path space, be such that it contains the polygonal paths, so that $P_n H \subset \Gamma$, for $n = 1, 2, \cdots$. The space H is not compact and so there is no Riesz-Markov theorem giving a 1–1 correspondence between regular Borel measures on H and positive functionals on the putative commutative Banach algebra of continuous functions on H.[15] The following way of obtaining a compact model for the path space Γ is due to Nelson.[4] *A priori*, it makes minimum smoothness assumptions on the paths. It leads to our construction of Wiener measure, which in part copies Nelson.

Let $\dot{\mathbb{R}}$ be the 1 point compactification of $\mathbb{R}$; then we put $\Gamma = \times_{0 \le \tau < t} \dot{\mathbb{R}}$, with the (weak) product topology. Tychonoff's theorem asserts that Γ is a compact Hausdorff space. The elements $\gamma \in \Gamma$ can be thought of as arbitrary functions $\gamma: [0, t) \to \mathbb{R} \cup \{\infty\}$. $C(\Gamma)$ denotes the Banach algebra of continuous functions defined on Γ. The Riesz–Markov theorem asserts that there is a 1–1 correspondence between positive functionals on $C_{\mathbb{R}}(\Gamma)$, the real-valued continuous functions defined on Γ, and regular Borel measures on Γ. This is basic in what follows.

We denote the space of functionals $\mathcal{J}(H) \cap C(\Gamma)$ by $\mathcal{J}_0$. Then we have the following lemma.

Lemma 1: $\mathcal{J}_0$ is dense in $C(\Gamma)$.

Proof: The proof is an easy application of the Stone–Weierstrass theorem.[15] Firstly $\mathcal{J}_0$ is closed under multiplication, because $M(H)$ is closed under convolution. Secondly $f \in \mathcal{J}_0 \Rightarrow \bar{f} \in \mathcal{J}_0$, the overbar being complex conjugate. Thirdly, if $\gamma \ne \gamma'$, $\gamma, \gamma' \in \Gamma$, $\exists\, \sigma \in [0, t)$ such that $\gamma(\sigma) \ne \gamma'(\sigma)$. Hence, $\exists\, \alpha \in \mathbb{R}$ with $\alpha[\gamma(\sigma) - \gamma'(\sigma)] \ne 0 \pmod{2\pi} \Rightarrow \exp[i\alpha\gamma(\sigma)] \ne \exp[i\alpha\gamma'(\sigma)]$. But $f[\gamma] = \exp[i\alpha\gamma(\sigma)] \in \mathcal{J}_0$ and so $\mathcal{J}_0$ separates points of Γ. Finally we have already seen that $1 \in \mathcal{J}_0$.

Denote by $\mathcal{J}_0^{-i}$, $\mathcal{J}_0^{-i} = \mathcal{J}^{-i} \,|\, \mathcal{J}_0$. Then we also have Lemma 2.

Lemma 2: $\mathcal{J}_0^{-i}$ has a continuous extension E to $C(\Gamma)$ such that, $\forall\, f \in C(\Gamma)$,

$$|E[f]| \le \|f\|_\infty = \sup_{\gamma \in \Gamma} |f[\gamma]|. \tag{26}$$

E is positive and so there is a unique regular Borel measure μ_w such that, $\forall\, f \in C(\Gamma)$,

$$E[f] = \int f[\gamma]\, d\mu_w(\gamma). \tag{27}$$

μ_w is just the Wiener measure with $\mathrm{supp}\,\mu_w \subset C_0(0, t)$, the space of continuous functions $\gamma: (0, t) \to \mathbb{R}$ with $\gamma(t) = 0$. *A posteriori*, $E = \mathcal{J}^{-i}$ and

$$|\mathcal{J}^{-i}[f]| \le \sup_{\gamma \in C_0(0,t)} |f[\gamma]|, \tag{28}$$

$\forall$ continuous functionals $f: C_0(0, t) \to \mathbb{C}$.

Proof: By definition $\mathcal{J}_0^{-i}$ is positive in the sense that $f \in \mathcal{J}_0$ with

$$f[\gamma] \ge 0, \; \forall\, \gamma \in \Gamma, \; \Rightarrow \mathcal{J}_0^{-i}[f] \ge 0.$$

Since $\mathcal{J}_0^{-i}[1] = 1$, it follows that, $\forall$ real-valued $f \in \mathcal{J}_0$,

$$\mathcal{J}_0^{-i}[f] \le \sup_{\gamma \in \Gamma} |f[\gamma]|. \tag{29}$$

Since $\mathrm{Re}\,\mathcal{J}_0^{-i}[f] = \mathcal{J}_0^{-i}[\mathrm{Re} f]$, we obtain for any $f \in \mathcal{J}_0$ (possibly complex-valued) with $\mathcal{J}^{-i}[f]=\exp(i\phi)r$, ϕ real and r real and positive,

$$|\mathcal{J}_0^{-i}[f]|=\mathrm{Re}\mathcal{J}_0^{-i}[\exp(-i\phi)f]=\mathcal{J}_0^{-i}[\mathrm{Re}[\exp(-i\phi)f]] \leq \sup_{\gamma\in\Gamma}|\mathrm{Re}[\exp(-i\phi)f[\gamma]]| \leq \sup_{\gamma\in\Gamma}|f[\gamma]|. \tag{30}$$

Hence $\mathcal{J}_0^{-i}$ has a unique positive extension E to the whole of $C(\Gamma)$. As Γ is compact, the Riesz–Markov theorem asserts that corresponding to E there is a unique regular Borel measure μ on Γ such that

$$E[f]=\int_\Gamma f[\gamma]\,d\mu(\gamma). \tag{31}$$

This proves the existence of the path-space measure μ in the case corresponding to $s=-i$. Of course, the same argument works for any negative imaginary value of s. As we shall see there is no path-space measure for other values of s.

One can now show in a number of different ways that $\mathrm{supp}\mu\subset C_0(0,t)$.[16] The fact that $\mu=\mu_w$, the usual Wiener measure with $\mathrm{supp}\mu_w\subset C_0(0,t)$, and that $E=\mathcal{J}^{-i}$ follows from the observation

$$\mathcal{J}_n^{-i}[f]=\int[f\circ P_n][\gamma]\,d\mu_w(\gamma) \tag{32}$$

and $f\in C(\Gamma)\Rightarrow(f\circ P_n)\to f$, a.e. w.r.t. Wiener measure μ_w, as $n\to\infty$. Letting $n\to\infty$ and using the dominated convergence theorem proves the result.

We have gone out of our way to show that $E=\mathcal{J}^{-i}$ is determined by its values on the apparently small space of functionals $\mathcal{J}_0$. Since it turns out that μ_w is supported by $C_0(0,t)$, the functionals in $\mathcal{J}_0$ need only be continuous on $C_0(0,t)$. We observe that if $f\in\mathcal{J}(H)$ is a continous functional $f:C_0(0,t)\to\mathbb{C}$, we have the Parseval identity

$$E[f]=\int_{C_0(0,t)}f[\gamma]\,d\mu_w(\gamma)=\int_H\exp(-\tfrac{1}{2}\|\gamma\|^2)\,d\mu_f(\gamma). \tag{33}$$

This seems to be a new result for Wiener integrals. It is basic to Theorem 4.

Theorem 4: Let $f\in\mathcal{J}(H)$; then $\mathcal{J}^s[f]$ is a regular analytic function of s in $\mathrm{Im}s<0$, continuous in $\mathrm{Im}s\leq 0$. If $f\in\mathcal{J}(H)$ is a continuous functional $f:C_0(0,t)\to\mathbb{C}$, then

$$\mathcal{J}[f]=\lim_{\epsilon\to 0+}\mathcal{J}^{1-i\epsilon}[f],\quad E[f]=\lim_{\epsilon\to 0}\mathcal{J}^{-i+\epsilon}[f]. \tag{34}$$

Interpolation gives

$$|\mathcal{J}^{se^{-i\alpha}}[f]|\leq\|f\|_0^{1-2\alpha/\pi}\|f\|_\infty^{2\alpha/\pi},$$
$$0\leq\alpha\leq\pi/2,\ s>0, \tag{35}$$

where $\|f\|_\infty=\sup_{\gamma\in C_0(0,t)}|f[\gamma]|$.

Proof: First, $\forall\,\gamma\in H$, $\exp[-(is/2)\|\gamma\|^2]\to\exp[-(is'/2)\|\gamma\|^2]$ as $s\to s'\in\mathbb{C}$ and $|\exp[-(is/2)\|\gamma\|^2]|\leq 1$, $\forall\,\gamma\in H$, $\mathrm{Im}s\leq 0$. Thus, the dominated convergence theorem for the measure μ_f in Theorem 3 gives the continuity result.

Let C be any simple closed contour in $\mathrm{Im}s<0$. Then, interchanging orders of integration by Fubini's theorem,

$$\oint_C\mathcal{J}_s[f]\,ds=\oint_C ds\int_H\exp[-(is/2)\|\gamma\|^2]\,d\mu_f(\gamma)$$
$$=\int_H d\mu_f(\gamma)\oint_C\exp[-(is/2)\|\gamma\|^2]\,ds=0. \tag{36}$$

Hence, for $f\in\mathcal{J}(H)$, Morera's theorem implies that $\mathcal{J}_s[f]$ is regular in $\mathrm{Im}s<0$.

Since $|\mathcal{J}_s[f]|\leq\|f\|_0$, when s is real, and $|\mathcal{J}_s[f]|\leq\|f\|_\infty$, when s is negative pure imaginary, the last part of the theorem follows by interpolation.

We showed in Lemma 2 that there was a path-space measure μ_w corresponding to $\mathcal{J}_s$ when s is a negative pure imaginary. The next lemma shows that this does not occur for any other value of s. The idea of the proof is, in essence, due to Cameron.[17]

Lemma 3: There is no path-space measure (of finite absolute variation) corresponding to $\mathcal{J}_s$ unless s is negative pure imaginary.

Proof: Let $s=s_1-is_2$, $s_2>0$, s_1 real. Assume that corresponding to $\mathcal{J}^s$ there is a path-space measure μ_s defined on Γ. For fixed n and $\mathrm{Im}a<0$, consider the functional $f_n:\Gamma\to\mathbb{C}$

$$f_n[\gamma]=\exp\left(\frac{-ia}{2\Delta t}\sum_{j=0}^{n-1}\Delta\gamma_j^2\right)$$
$$=(2\pi ia\Delta t)^{-n/2}\int d^n\gamma'\exp\left(i\sum_{j=0}^{n-1}\Delta\gamma_j'\Delta\gamma_j\Delta t^{-1}+\frac{i}{2a}\sum_{j=0}^{n-1}\Delta\gamma_j'^2\Delta t^{-1}\right), \tag{37}$$

where $\Delta\gamma_j=\gamma_{j+1}-\gamma_j$ and $\gamma_j=\gamma(jt/n)$, $j=0,1,2,\ldots,n$. We assert that, for $\mathrm{Im}a<0$, $f_n\in\mathcal{J}(H)$. To see this, let $\pi_n:\mathbb{R}^n\to H$ be defined by

$$\pi_n(\gamma_0,\ldots,\gamma_{n-1})=\sum_{j=0}^{n-1}\left[G\left(\frac{j+1t}{n},\cdot\right)-G\left(\frac{jt}{n},\cdot\right)\right]\Delta\gamma_j\Delta t^{-1}. \tag{38}$$

Define the measure ν on $\mathbb{R}^n$ by

$$\nu(B)=(2\pi ia\Delta t)^{-n/2}\int_B d^n\gamma'\exp\left(\frac{i}{2a}\sum_{j=0}^{n-1}\Delta\gamma_j'^2\Delta t^{-1}\right), \tag{39}$$

for any Borel set $B\subset\mathbb{R}^n$. Then, for $\mathrm{Im}a<0$, $\mu_{f_n}\in M(H)$ is given by

$$\mu_{f_n}(A)=\nu(\pi_n^{-1}A), \tag{40}$$

for any Borel set $A\subset H$, and

$$f_n[\gamma]=\int\exp[i(\gamma',\gamma)]\,d\mu_{f_n}(\gamma'). \tag{41}$$

Then, from Theorem 3 and the hypothesis we obtain

$$\int f_n[\gamma]\,d\mu_s(\gamma)=\mathcal{J}^s[f_n]$$
$$=\int\exp\left(\frac{-is}{2}\|\gamma\|^2\right)d\mu_{f_n}(\gamma)$$
$$=(2\pi ia\Delta t)^{-n/2}\int\exp\left(\frac{-is}{2}\Sigma\Delta\gamma_j^2\Delta t^{-1}+\frac{i}{2a}\Sigma\Delta\gamma_j^2\Delta t^{-1}\right)d^n\gamma. \tag{42}$$

Explicitly calculating the integral gives

$$\int f_n[\gamma]\,d\mu_s(\gamma)=(1-as)^{-n/2}. \tag{43}$$

We now put $a=(s_1-i\epsilon)/|s|^2$, so $\mathrm{Im}a<0$ when $\epsilon>0$.

Then $|(1-as)^{-1}| = |s|/(s_2+\epsilon)$. When $s_1 \neq 0$, we can choose $\epsilon > 0$ sufficiently small so that $\rho = |(1-as)^{-1}| > 1$. Since $\sup_{\gamma\in\Gamma}|f_n[\gamma]| \leq 1$, we arrive at

$$\int_\Gamma d|\mu_s| \geq \left|\int_\Gamma f_n[\gamma]\,d\mu_s(\gamma)\right| = \rho^{n/2}, \quad \text{with } \rho > 1, \tag{44}$$

and n can be made arbitrarily large. Thus, when $s_1 \neq 0$, μ_s must have *infinite* absolute variation, proving the result.

We reiterate here that there is no need to assume that the path space Γ is the Hilbert space H. All that is required by our definition is that the path space $\Gamma \supset P_n H$, $n = 1, 2, \cdots$. We remark here that for the Wiener integral, when $\Gamma = C_0(0,t)$, the Hilbert space H has measure zero. This partly overcomes our earlier objections to the paths in H being too smooth. It is a general feature of abstract Wiener spaces[18] that the analog of H has canonical Gaussian measure zero. In the future to simplify domain considerations we shall assume $\Gamma = C_0(0,t)$ or $\Gamma = H$, it being clear from the context what the appropriate path space should be.

4. TRANSLATION AND CAMERON-MARTIN FORMULAS FOR FEYNMAN MAPS

The translation formula giving the transformation law for Wiener integrals under the change of integration variables $\gamma \to \gamma + a$, fixed $a \in H$, and the corresponding Cameron—Martin formula giving the transformation law for Wiener integrals under the linear change of variables $\gamma \to (1+K)\gamma$, with $KC_0(0,t) \subset H$, $(1+K)|_H$ being a linear injection and K trace class, are well known.[19] In this section we establish the corresponding results for the Feynman maps. This, incidentally, enables us to prove in Corollary 3 that the class of integrable functionals $\mathcal{J}^s(P_\infty H)$ includes a much wider class of functionals than $\mathcal{J}(H)$. We shall show in a future paper that the new integrable functionals have important applications to nonrelativistic quantum mechanics. Our results include as a special case the Cameron—Martin and translation formulas for Wiener integrals.

The translation formula is the content of Theorem 5.

Theorem 5: Let $a \in H$ and denote by $g_a : H \to \mathbb{C}$, $g_a[\gamma] = g[\gamma + a]$, where $g[\cdot] \in \mathcal{J}(H)$. Then, if $\operatorname{Im} s \leq 0$,

$$\mathcal{J}_s[\exp[(i/s)(a,\cdot)]g_a[\cdot]] = \exp[-(i/2s)\|a\|^2]\mathcal{J}_s[g]. \tag{45}$$

Proof: By definition we have

$$\begin{aligned}\mathcal{J}_s&[\exp[(i/s)(a,\cdot)]g_a[\cdot]]\\ &= \lim_{n\to\infty}\mathcal{J}_n^s[\exp[(i/s)(a,\cdot)]g_a[\cdot]]\\ &= \lim_{n\to\infty} N_n \int \exp[(i/2s)(\gamma, P_n\gamma) + (i/s)(a, P_n\gamma)]g_a[P_n\gamma]d^n\gamma,\end{aligned} \tag{46}$$

where $N_n = (2\pi i s\Delta t)^{-n/2}$, $d^n\gamma = d\gamma_0 \cdots d\gamma_{n-1}$ and each integration is from $-\infty$ to $+\infty$. Observing that $\gamma_j = -\sum_{r=j}^{n-1} \times \Delta\gamma_r$, we change integration variables to $\Delta\gamma_j = \gamma_{j+1} - \gamma_j$, $j = 0, 1, 2, \ldots, n-1$. We now simply complete the square in the exponential to give, with $f[\cdot] = \exp[(i/s)(a,\cdot)] \times g_a[\cdot]$,

$$\begin{aligned}\mathcal{J}_s[f] = \lim_{n\to\infty} N_n \int &\exp[(i/2s)\|P_n(\gamma + a)\|^2]g_a[P_n\gamma]\\ &\times d(P_n\gamma)\exp[-(i/2s)\|P_n a\|^2],\end{aligned} \tag{47}$$

where $d(P_n\gamma) = d\Delta\gamma_0\, d\Delta\gamma_1 \cdots d\Delta\gamma_{n-1}$. Changing integration variables now from $d(P_n\gamma)$ to $d(P_n\gamma + a)$, in an obvious notation with $\gamma' = \gamma + a$,

$$\begin{aligned}\mathcal{J}_s[f] = \lim_{n\to\infty} N_n \int &\exp[(i/2s)(\gamma', P_n\gamma')]g_a[P_n\gamma' - P_n a]\\ &\times d(P_n\gamma')\exp[-(i/2s)(a, P_n a)].\end{aligned} \tag{48}$$

Writing $h_n[\gamma] = g_a[\gamma - P_n a] = g[\gamma + (1-P_n)a]$, we see that $h_n[\cdot] \in \mathcal{J}(H)$ and

$$h_n[\gamma] = \int \exp[i(\gamma', \gamma)]\exp[i(\gamma', \overline{1-P_n}a)]\,d\mu_g(\gamma'). \tag{49}$$

Hence, from the first part of Theorem 3 we obtain

$$\begin{aligned}\mathcal{J}_s[f] &= \lim_{n\to\infty}\exp[-(i/2s)(a, P_n a)]\mathcal{J}_n^s[h_n]\\ &= \lim_{n\to\infty}\exp[-(i/2s)(a, P_n a)]\int\exp[-(is/2)(\gamma, P_n\gamma)]\\ &\qquad\times\exp[i(\gamma, \overline{1-P_n}a)]\,d\mu_g(\gamma).\end{aligned} \tag{50}$$

The result follows from Theorem 2 and the dominated convergence theorem for μ_g.

Theorem 5 has a corollary:

Corollary 2: Let E denote expectation w.r.t. the Wiener measure μ_w. Then, for fixed $a \in H$, with $\dot{a}$ of bounded absolute variation,

$$E[\exp[-(a,\cdot)]g_a[\cdot]] = \exp(\|a\|^2/2)E[g], \tag{51}$$

when $g : C_0(0,t) \to \mathbb{C}$ is a continuous bounded functional.

Proof: Firstly when $\dot{a}$ is of bounded absolute variation $(a,\cdot)$ is defined a.e. w.r.t. Wiener measure μ_w as a Stieltjes integral. If $a \in H$ is such that $\dot{a}$ is of bounded absolute variation and g is a continuous bounded functional integrand on the lhs is a continuous bounded functional mapping $C_0(0,t) \to \mathbb{C}$. Putting $s = -i$ in Theorem 5, with $\dot{a}$ of bounded absolute variation, the above result is true for $g \in \mathcal{J}_0$. As $\mathcal{J}_0$ is dense in $C(\Gamma)$, the result follows.

The Cameron—Martin formula is the content of the next theorem.

Theorem 6: Let $(1+K) : H \to H$ be a linear injection with K trace class and $\det(1+K) \neq 0$. Let $g : H \to \mathbb{C}$ and define $g_{1+K} : H \to \mathbb{C}$ by $g_{1+K}[\gamma] = g[(1+K)\gamma]$. It is convenient to denote by $e_s^K g_{1+K}$

$$[e_s^K g_{1+K}][\gamma] = \exp[(i/s)(K\gamma, \gamma) + (i/2s)(K\gamma, K\gamma)]g_{1+K}[\gamma], \tag{52}$$

so that $(e_s^K g_{1+K}) : H \to \mathbb{C}$. Then, if $g \in \mathcal{J}^s(P_\infty H)$, $(e_s^K g_{1+K}) \in \mathcal{J}^s(P_\infty H)$, and

$$\mathcal{J}_s[e_s^K g_{1+K}] = |\det(1+K)|^{-1}\mathcal{J}_s[g]. \tag{53}$$

Proof: By definition we have

$$\begin{aligned}\mathcal{J}_s[e_s^K g_{1+K}] &= \lim_{n\to\infty}\mathcal{J}_n^s[e_s^K g_{1+K}]\\ &= \lim_{n\to\infty} N_n \int \exp[(i/2s)\|(1+K)P_n\gamma\|^2]\\ &\qquad\times g_{1+K}[P_n\gamma]d(P_n\gamma).\end{aligned} \tag{54}$$

In this integral the variables of integration $d(P_n\gamma)$ denote

$$d\Delta\gamma_j = d[\gamma(\overline{j+1}\,t/n) - \gamma(jt/n)], \quad j=0,1,2,\ldots,n-1.$$

We shall change integration variables to $\Delta\gamma_j' = [\gamma'(\overline{j+1}\,t/n) - \gamma'(jt/n)]$, $j=0,1,2,\ldots,n-1$, where

$$\gamma' = (1+K)P_n\gamma. \tag{55}$$

To carry out this change of integration variables, we must calculate the Jacobian determinant by using the reproducing kernel.

We denote by $e_j^n(\cdot) = [G(\overline{j+1}\,t/n, \cdot) - G(jt/n, \cdot)](n/t)^{1/2}$, $j=0,1,2,\ldots,n-1$. It is simple to check from the reproducing property that $e_0^n, e_1^n, \ldots, e_{n-1}^n$ is an orthonormal system of vectors in H. (We return to this point later.)

We have

$$P_n\gamma = \sum_{j=0}^{n-1} (\Delta\gamma_j)\left(\frac{n}{t}\right)^{1/2} e_j^n \tag{56}$$

and from reproducing property, for $k=0,1,2,\ldots,n-1$,

$$\Delta\gamma_k'\left(\frac{n}{t}\right)^{1/2} = (e_k^n, \gamma') = \left(e_k^n, \sum_{j=0}^{n-1}(1+K)\Delta\gamma_j\left(\frac{n}{t}\right)^{1/2} e_j^n\right)$$
$$= \sum_{j=0}^{n-1}\Delta\gamma_j\left(\frac{n}{t}\right)^{1/2}(e_k^n, (1+K)e_j^n). \tag{57}$$

If we denote the $(n\times n)$ identity by I_n, the Jacobian determinant for the change of integration variables is just

$$\frac{d(P_n\gamma)}{d(P_n\gamma')} = \det[I_n + P_nKP_n]^{-1} = |J^n|^{-1}. \tag{58}$$

But $P_n \xrightarrow{s} I$ and so $|J^n| \to \det(1+K) \neq 0$, as $n\to\infty$. Thus, for sufficiently large n, $|J^n| \neq 0$, and we can change the variables of integration to $d(P_n\gamma')$.

The circumstance $|J^n| \neq 0$ is also just the condition for $(1-P_n)KP_n = 0$. To see this, we write $(1+K)P_nH$ as an orthogonal sum

$$(1+K)P_nH = P_n(1+K)P_nH \oplus (1-P_n)(1+K)P_nH. \tag{59}$$

Evidently $\dim\{(1+K)P_nH\} \leq \dim(P_nH) = n$. We now show that, if $J_n \neq 0$, $\dim\{P_n(1+K)P_nH\} = n$. This is a simple consequence of Eq. (57).

Let $P_n\gamma' \in P_n(1+K)P_nH$, where $\gamma' = (1+K)P_n\gamma$, $\gamma\in H$. Then we can write

$$P_n\gamma' = \sum_{j=0}^{n-1}(e_j^n, P_n\gamma')e_j^n = \sum_{j=0}^{n-1}(e_j^n, \gamma')e_j^n$$
$$= \sum_{j=0}^{n-1}\Delta\gamma_j'\left(\frac{n}{t}\right)^{1/2} e_j^n, \tag{60}$$

where we are using $(e_j^n, e_k^n) = \delta_{jk}$, $j,k = 0,1,2,\ldots,n-1$, $P_n^* = P_n$, and $P_ne_j = e_j$, $j=0,1,2,\ldots,n-1$.

However, we have already seen that

$$\Delta\gamma_j' = \sum_{l=0}^{n-1} J_{jl}^n\Delta\gamma_l, \quad j=0,1,\ldots,n-1, \tag{61}$$

where $P_n\gamma = \sum_{l=0}^{n-1}\Delta\gamma_l(n/t)^{1/2}e_l$.

We now fix m and choose $P_n\gamma$ so that $\Delta\gamma_l = (J^n)^{-1}_{lm}$, $l=0,1,\ldots,n-1$. Then we obtain from above $\Delta\gamma_j' = \delta_{jm}$, $j=0,1,2,\ldots,n-1$, or $P_n\gamma' = (n/t)^{1/2}e_m$. Letting $m=0,1,\ldots,n-1$, in turn, proves that if $|J^n| \neq 0$, $\dim[P_n(1+K)P_n] = n$ and $(1-P_n)KP_n = 0$.

Thus far we have seen that $\det(1+K)\neq 0 \Rightarrow |J^n| \neq 0$, for sufficiently large n, $\Rightarrow (1-P_n)KP_n = 0$. Choosing n sufficiently large so that $|J^n| \neq 0$ and changing integration variables to $d(P_n\gamma')$, we obtain

$$\mathcal{J}_s^n[e_s^K g_{1+K}] = |J_n^{-1}| N_n \int \exp\{(i/2s)\|P_n\gamma'\|^2\}$$
$$\times g[P_n\gamma']d(P_n\gamma'). \tag{62}$$

Letting $n\to\infty$, the result follows.

If we accept one or two probabilistic technicalities, we can prove the Cameron–Martin formula for Wiener integrals.

Corollary 3: Let $(1+K): C_0(0,t)\to C_0(0,t)$ with $K[C_0(0,t)]\subset H$. Let $(1+K)|_H : H\to H$ be a linear injection with $K|_H$ trace class and $\det(1+K)\neq 0$. Let $g: C_0(0,t)\to\mathbb{C}$ be a bounded continuous functional and denote by E the expectation w.r.t. Wiener measure μ_w. Then

$$E[\exp[-(K\cdot,\cdot) - \tfrac{1}{2}(K\cdot, K\cdot)]g_{1+K}[\cdot]]$$
$$= [\det(1+K)]^{-1}E[g]. \tag{63}$$

Proof: Here $(K\cdot,\cdot)$ is defined as a random variable according to Kuo and $(KP_n\cdot, P_n\cdot)\to(K\cdot,\cdot)$ in probability as $P_n\xrightarrow{s} I$, because $P_nH\subset C_0^*(0,t)$, $n=1,2,\cdots$ (see Ref. 18, p. 142). The proof follows as in Corollary 2.

The next corollary has important applications to quantum mechanics.

Corollary 4:

$$\mathcal{J}^s(P_\infty H) \supset \bigcup_K e_s^K[\mathcal{J}(H)]_{1+K} \quad (\mathrm{Im}\, s \leq 0,\; s\neq 0), \tag{64}$$

where the union is over all trace class K with $(1+K): H\to H$ an injection, $\det(1+K)\neq 0$.

Proof: The result follows from the last theorem and the fact that $\mathcal{J}(H)\subset\mathcal{J}^s(P_\infty H)$, when $\mathrm{Im}\, s\leq 0$.

To see the power of the last corollary, we observe that $1\in\mathcal{J}(H)$. This enables us to integrate an enormous number of quadratic exponentials occurring in nonrelativistic quantum mechanics. We shall discuss some applications of these results in a future publication.

5. CONCLUSION

In this paper we have shown that the Feynman path integral $\mathcal{J}^1$ can be obtained as the analytic continuation of the Wiener integral $\mathcal{J}^{-i}$ by introducing the Feynman maps $\mathcal{J}^s$. We have seen that a number of basic properties of the Wiener integral generalise for the Feynman maps $\mathcal{J}^s$. In particular we have seen that the Feynman maps $\mathcal{J}^s$ have analogs of the translation and Cameron–Martin formulas. On the negative side we have shown that there is no path-space measure corresponding to the Feynman maps $\mathcal{J}^s$, when $\mathrm{Re}\, s\neq 0$. This makes it rather difficult to prove any sort of dominated convergence theorem for the Feynman path integral $\mathcal{J}^1$. There is, however, a rather weak dominated convergence theorem for $\mathcal{J}^1$, which we include here for the sake of completeness. In some ways this theorem is an extension of Theorem 4.

Theorem 7: Let $\{f_n\}$ be a sequence of continuous functionals, $f_n \in \mathcal{J}(H)$, $f_n : C_0(0, t) \to \mathbb{C}$, uniformly bounded in $\| \|_0$. Let f be a continuous functional such that $g(-is) = \mathcal{J}_{-is}[f]$, $s > 0$, is the restriction to the negative imaginary axis of the function $g(s)$, which is regular analytic[23] in $\text{Im}s < 0$, with $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Then, if f is such that $(f \circ P_n) \in \mathcal{J}(H)$ and $\{f \circ P_n\}$ is uniformly bounded in $\| \|_0$, for a.e. real s, $f \in \mathcal{J}^s(P_\infty H)$ and

$$\mathcal{J}_s[f] = \lim_{n\to\infty} \mathcal{J}_s[f_n] = g(s), \tag{65}$$

i.e., with probability one the Feynman integral of f is equal to the analytic continuation of the Wiener integral of f and the Feynman integral of the limit of a sequence of functions uniformly bounded in $\| \|_0$ is equal to the limit of the corresponding sequence of Feynman integrals.

The proof depends upon two elementary lemmas.

Lemma 4: Let $\{f_n\}$ be a sequence of continuous functionals $f_n \in \mathcal{J}(H)$ uniformly bounded in $\| \|_0$. Let $\|f_n - f\|_\infty \to 0$, as $n \to \infty$, where f is a continuous functional with the property that $g(-is) = \mathcal{J}_{-is}[f]$, $s > 0$, is the restriction to the negative imaginary axis of the function $g(s)$, which is regular analytic in $\text{Im}s < 0$. Then, if the limit exists, for a.e. $s > 0$,

$$\lim_{n\to\infty} \mathcal{J}_s[f_n] = g(s). \tag{66}$$

Proof: $\{h_n \mid h_n(s) = \mathcal{J}_s[f_n]\}$ is a uniformly bounded sequence of regular analytic functions in the domain $\text{Im}s < 0$. Hence, $\{h_n\}$ is a normal family of regular analytic functions.[20] Thus, as $\{s \mid s = -is', \ s' > 0\}$ is a determining set in $\text{Im}s < 0$, $h_\infty(s) = \lim_{n\to\infty} \mathcal{J}_s[f_n]$ exists $\forall \ \text{Im}s < 0$ and is a bounded regular function of s in this region. However, $h_\infty(s) = g(s)$, $s = -is'$, $s' > 0 \Rightarrow h_\infty(s) = g(s)$, $\forall \ \text{Im}s < 0$. The Fatou–Privaloff theorem[21] for $h_\infty(s) \Rightarrow$ if the limit exists, for a.e. $s > 0$, $\lim_{n\to\infty} \mathcal{J}_s[f_n] = g(s)$.

Lemma 5: Let f be a continuous functional $f : C_0(0, t) \to \mathbb{C}$ such that $g(-is) = \mathcal{J}_{-is}[f]$, $s > 0$, is the restriction to the negative imaginary axis of the function $g(s)$, which is regular analytic in $\text{Im}s < 0$. Then, if $(f \circ P_n) \in \mathcal{J}(H)$, for each n, and $\{f \circ P_n\}$ is uniformly bounded in $\| \|_0$, for a.e. $s > 0$, $f \in \mathcal{J}^s(P_\infty H)$ and

$$\mathcal{J}_s[f] = \lim_{n\to\infty} \mathcal{J}_s[f \circ P_n] = g(s). \tag{67}$$

Proof: First $(f \circ P_n) \in \mathcal{J}(H) \Rightarrow \mathcal{J}_n^s[f] = \mathcal{J}_s[f \circ P_n]$. To see this, we let $\mu_{f \circ P_n} = \mu_n \in M(H)$. Then $[f \circ P_n][\gamma] = \int \exp[i(\gamma', \gamma)]\, d\mu_n(\gamma')$. Since $P_n^2 = P_n$, we obtain $[f \circ P_n] \times [\gamma] = \int \exp[i(\gamma', P_n\gamma)]\, d\mu_n(\gamma')$. Hence, $[f \circ P_n][\gamma] = \int \exp[i(\gamma'', \gamma)]\, d\mu_n(P_n^{-1}\gamma'') \Rightarrow \mathcal{J}_s[f \circ P_n] = \int \exp[-(is/2) \times \|\gamma\|^2]\, d\mu_n(P_n^{-1}\gamma)$.

From the first part of Theorem 3 we obtain[22]

$$\mathcal{J}_s[f \circ P_n] = \int \exp[-(is/2)(\gamma', P_n\gamma')]\, d\mu_n(\gamma')$$
$$= \mathcal{J}_n^s[f]. \tag{68}$$

Thus, $\{f_n\} = \{f \circ P_n\}$ satisfies the conditions of the last lemma and so, for a.e. $s > 0$,

$$g(s) = \lim_{n\to\infty} \mathcal{J}_s[f \circ P_n] = \lim_{n\to\infty} \mathcal{J}_n^s[f] = \mathcal{J}_s[f], \tag{69}$$

proving the result.

To prove Theorem 7, we now simply combine Lemmas 5 and 6 to give, for a.e. $s > 0$,

$$\lim_{n\to\infty} \mathcal{J}_s[f_n] = g(s) = \mathcal{J}_s[f], \tag{70}$$

so, with probability one, the Feynman integral is the analytic continuation of the Wiener integral, even when $f \notin \mathcal{J}(H)$. Theorem 7 partly explains the many tantalizing identities between the Feynman and Wiener integrals. When $f \in \mathcal{J}(H)$, it can be considered as a special case of Theorem 4.

The above work and the results of previous papers suggest that $\mathcal{J} = \mathcal{J}^1$ is a good candidate for a workable definition of the Feynman path integral in nonrelativistic quantum mechanics. Certainly it is easier to work with $\mathcal{J}$ than with most previous definitions and, as we have seen, it enables us to integrate a fairly wide class of functionals. Associated with $\mathcal{J}$ there is a sufficiently substantial body of theorems to make it a reliable and efficient calculational tool. We shall discuss some applications of the above results in a future paper. Before concluding this section, however, we wish to indicate a connection between some of our ideas and some elementary probabilistic notions.

It is clear that in the above we have made repeated use of the fact that $P_n : H \to H$ is a projection. This fact is due to the property that $\{e_0^n, e_1^n, \ldots, e_{n-1}^n\}$ is an orthonormal system in H. There is an *a priori* probabilistic reason for this to be so—namely that the Wiener process had independent normally distributed random increments $\Delta\gamma_j$ with variance Δt.

It is a simple matter to deduce this orthonormality of $\{e_0^n, e_1^n, \ldots, e_{n-1}^n\}$ from the well-known property of Wiener measure that $\forall \ a, b \in H$,

$$E[(a, \gamma)(b, \gamma)] = (a, b), \tag{71}$$

by putting $a = e_j^n$, $b = e_k^n$, using the reproducing kernel property and the independence of the normally distributed $\Delta\gamma_j$. It is interesting to note that, when we put $a(\cdot) = G(\sigma, \cdot)$, $b(\cdot) = G(\tau, \cdot)$, we arrive at the intriguing identity for the reproducing kernel

$$E[\gamma(\sigma)\gamma(\tau)] = G(\sigma, \tau), \tag{72}$$

$\sigma, \tau \in [0, t]$.

Thus, there are *a priori* probabilistic reasons why Feynman's polygonal paths are so well-suited for defining a path integral. We feel that, bearing in mind Feynman's original ideas on the path integral, our definition of the path integral is the most natural to make. It is reassuring that, with this definition, despite the absence of a path-space measure most results of the Wiener integral generalize quite easily by using the reproducing kernel.

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[22]A more direct way of proving this result is to use the Chapman—Kolmogorov equations:

$$\int_{-\infty}^{\infty} d\gamma_k [2\pi i s (t_{k+1} - t_k)]^{-1/2} \exp\left(\frac{i(\gamma_{k+1} - \gamma_k)^2}{2s(t_{k+1} - t_k)}\right) \times [2\pi i s (t_k - t_{k-1})]^{-1/2} \exp\left(\frac{i(\gamma_k - \gamma_{k-1})^2}{2s(t_k - t_{k-1})}\right) = [2\pi i s (t_{k+1} - t_{k-1})]^{-1/2} \exp\left(\frac{i(\gamma_{k+1} - \gamma_{k-1})^2}{2s(t_{k+1} - t_{k-1})}\right), \quad \mathrm{Im}\, s \leqslant 0,$$

with $t_{k+1} > t_k > t_{k-1}$. The details of the argument, however, are more complicated.

[23]This will be the case if and only if $f_n(\cdot) \to f(\cdot)$ a.e. w.r.t. Weiner measure, as $n \to \infty$.

The theory of superselection rules. I. A class of inequivalent, irreducible *-representations of the canonical commutation relations of the free electromagnetic field

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We begin here the rigorous construction of new superselection sectors for the free quantum electromagnetic field by exhibiting a wide class of inequivalent irreducible *-representations of the canonical commutation relations of the electromagnetic field. The *-representations constructed here satisfy all the axioms of Haag and Kastler, except possibly Poincaré covariance. In a forthcoming paper, the construction of the new superselection sectors is completed with the study of the spectrum condition for the *-representations.

1. INTRODUCTION

This paper is the first part of a two part comprehensive study of the theory of superselection rules for the free quantum electromagnetic field,[1] along the C^*-algebraic lines of Haag and Kastler[2] and of Doplicher, Haag, and Roberts.[3–5] In this first part, we construct a wide class of inequivalent, irreducible $*$-representations of the canonical commutation relations of the free quantum electromagnetic field. The $*$-representations satisfy all the axioms of Ref. 2, except possibly Poincare covariance. It is well known[6] that for the free quantum electromagnetic field, because of infrared problems, Poincare covariant $*$-representations are hard to come by. But it is, of course, desirable to know under what conditions each of the $*$-representations satisfies the spectrum condition. In Ref. 7, we study the conditions under which space–time automorphisms of the C^*-algebra of quasilocal observables for each of the $*$-representations are implemented by a strongly continuous unitary representation of Minkowski space, an additive group, such that the infinitesimal generator, the energy–momentum operator, of the unitary representation has spectrum in the closed forward light cone. Such a $*$-representation, in the terminology of Borchers,[8] is called positive. A superselection sector is an equivalence class of positive $*$-representations, and each sector carries labels called superselection quantum numbers. The exercise is to provide a construction which predicts all the superselection quantum numbers. The sectors whose construction we begin here and complete in Ref. 7 are labeled by continuous real numbers. In the two-dimensional (space–time) models of Streater and Wilde[9] and of Bonnard and Streater,[10] the superselection sectors have also been found to be labeled by continuous real numbers. This explains why the recent work of Frohlich[11] which gives models of interacting fields, also in two dimensions, whose superselection sectors are labeled by nonnegative integers, is of much interest. However, it has also been pointed out in Ref. 11 that the construction undertaken in Ref. 11 would fail in four-dimensional Minkowski space, where we carry out our own analysis, except possibly for non-Abelian Yang–Mills theories. Our construction leans heavily on the theory of simplectic transformations due to Segal.[12,13]

2. FOCK SPACE AND THE FREE QUANTUM ELECTROMAGNETIC FIELD

Let $\mathcal{R}^4$ be the 4-fold Cartesian product of $\mathcal{R}$, the real line, with itself. If convenient, we sometimes consider $\mathcal{R}^4$ as represented in the form $\mathcal{R}^4=\mathcal{R}\times\mathcal{R}^3$. Then, each $x\in\mathcal{R}^4$ is of the form $x=(x_0,\mathbf{x})$ with $x_0\in\mathcal{R}$ and $\mathbf{x}\in\mathcal{R}^3$.

Let M^4 denote Minkowski space. Then M^4 is the couple $(\mathcal{R}^4, [\cdot,\cdot])$ where $[\cdot,\cdot]$ is the bilinear mapping:

$$[\cdot,\cdot]:\mathcal{R}^4\times\mathcal{R}^4\to\mathcal{R}$$
$$(x,y)\mapsto [x,y]=x_0y_0-\sum_{i=1}^{3}x_iy_i.$$

The bilinear pairing $[\cdot,\cdot]$ is an indefinite inner product on M^4. Let $(M^4)^*$ denote the topological dual of M^4 and let $V\subset(M^4)^*$ be the cone

$$V=\{p=(p_0,\mathbf{p})\in(M^4)^*:\ [p,p]=0 \text{ with } p_0>0\}.$$

It is well known that the Lorentz-invariant measure on V is

$$m(dp)=\frac{1}{p_0}\,d\mathbf{p}=\frac{1}{|\mathbf{p}|}\,d\mathbf{p}.$$

In the displayed expression for $m(dp)$, we have implicitly chosen $\mathbf{p}=(p_1,p_2,p_3)$ as a global coordinate system for V.

Let $L^0(V)$ be the equivalence class of all Lebesgue measurable complex-valued four-component vector functions $f=(f_0,f_1,f_2,f_3)$ on V which satisfy the following four conditions:

(i) $f(-p)=\bar{f}(p)$ (reality condition),

(ii) $[p,f(p)]=p_0f(p)-\sum_{i=1}^{3}p_if_i(p)=0$,

(Lorentz gauge condition)

(iii) $\|f\|^2=\int_V m(dp)[f(p),\bar{f}(p)]<\infty$,

(iv) $\int_V m(dp)\sum_{i=0}^{3}|f_i(p)|^2<\infty$.

In view of (iv), for $f\in L^0(V)$

$$h(x)=\int_V m(dp)f(p)\,e^{-i[p,x]},\quad x\in\mathcal{R}^4,$$

exists as a four-component vector tempered distribution on $\mathcal{R}^4$.

The quotient space $L^0(V)/(\text{Kernel}\,\|\cdot\|)$ is a real

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Hilbert space $\mathcal{J}_1'$ with norm $\|\cdot\|_{\mathcal{J}_1'} = \|\cdot\|$. Let $\mathcal{J}_1$ be the complexification of $\mathcal{J}_1'$ and set $\mathcal{J}_0 = \mathbb{C}$, the complex numbers. Let $\mathcal{J}_n$, $n=1, 2, \cdots$, denote the n-fold symmetric tensor product of $\mathcal{J}_1$ with itself. Then the Hilbert space

$$\mathcal{J} = \bigoplus_{n=0} \mathcal{J}_n$$

is the Fock space for the free quantum electromagnetic field, which we introduce next.

Let $F \in L^2(\mathcal{R}^3, d\mathbf{p})$ and $\Psi \in \mathcal{J}$. Then

$$a_\mu(F): \mathcal{J}_n \to \mathcal{J}_{n-1}$$

is the operator

$$(a_\mu(F)\Psi)^{n-1}{}_{\mu_1\mu_2,\cdots,\mu_{n-1}}(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_{n-1})$$

$$= \sqrt{n} \int d\mathbf{p}\, F(\mathbf{p})\, \Psi^{(n)}{}_{\mu\, \mu_1\mu_2\cdots\mu_{n-1}}(\mathbf{p}, \mathbf{p}_1, \cdots, \mathbf{p}_{n-1}).$$

The operator $a_\mu(F)$ annihilates the vacuum vector in $\mathcal{J}$.

Next, $a_\mu^*(F): \mathcal{J}_n \to \mathcal{J}_{n+1}$ is the operator

$$(a_\mu^*(F)\Psi)^{(n)}{}_{\mu_1\mu_2\cdots\mu_n}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n+1})$$

$$= \frac{(-1)}{\sqrt{n+1}} \sum_{j=1}^{n+1} g_{\mu\,\mu_j} |\mathbf{p}_j| F(\mathbf{p}_j)$$

$$\times \Psi^{(n)}{}_{\mu_1\mu_2\ldots\hat{\mu}_j\ldots\mu_{n+1}}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \hat{\mathbf{p}}_j, \ldots, \mathbf{p}_{n+1}),$$

where $g_{\mu\nu}$ is the usual Minkowskian metric tensor with signature $(1,-1,-1,-1)$ and $\mu, \nu = 0,1,2,3$.

Each of the operators $a_\mu(F)$ and $a_\mu^*(F)$ is unbounded and has $\mathcal{J}_{\text{alg}}$, the algebraic direct sum of the $\mathcal{J}_n$, as dense domain in $\mathcal{J}$.

Before defining the field operator Φ_μ and its canonical conjugate π_μ, $\mu = 0,1,2,3$, recall that the potential Φ_μ is neither unique nor observable but the field strength $\Phi_{\mu\nu} = \partial\Phi_\nu/\partial x_\mu - \partial\Phi_\mu/\partial x_\nu$ is. Therefore, since this is a theory of observables, we are constrained to employ a specially defined space of functions as index space for Φ_μ.

Let $\mathcal{D}(\mathcal{R}^3)$ be the real Schwartz space of C^∞ functions with compact support in $\mathcal{R}^3$, and let $\mathcal{D}^4(\mathcal{R}^3)$ denote the 4-fold Cartesian product of $\mathcal{D}(\mathcal{R}^3)$ with itself. Let $\mathcal{D}_0^4(\mathcal{R}^3)$ be the subspace of $\mathcal{D}^4(\mathcal{R}^3)$ consisting of all $f = (f_0, f_1, f_2, f_3) \in \mathcal{D}^4(\mathcal{R}^3)$ such that each component f_μ of f is of the form

$$f_\mu = \sum_{\nu=0}^{3} \frac{\partial f_{\mu\nu}}{\partial x_\nu}, \quad \mu = 0,1,2,3,$$

where $f_{\mu\nu} \in \mathcal{D}(\mathcal{R}^3)$ and $f_{\mu\nu} + f_{\nu\mu} = 0$. Each $f \in \mathcal{D}_0^4(\mathcal{R}^3)$ satisfies $\tilde{f}(0) = 0$, where $\tilde{f}$ is the Fourier transform of f. Let Δ denote the Laplace operator in three variables and set $(-\Delta)^{1/4} = C$.

Let $\mathcal{H}^*$ be the completion of $\mathcal{D}_0^4(\mathcal{R}^3)$ in the topology derived from the norm

$$\|\cdot\|_{\mathcal{H}^*}: \mathcal{D}_0^4(\mathcal{R}^3) \to \mathcal{R}_+ = [0, \infty)$$

$$f = (f_0, f_1, f_2, f_3) \to \|f\|_{\mathcal{H}^*}$$

$$= [\sum_{\mu=0}^{3} \int d\mathbf{x}\, |(C^{-1}f_\mu)(\mathbf{x})|^2]^{1/2}$$

Similarly, $\mathcal{H}$ will denote the completion of $\mathcal{D}_0^4(\mathcal{R}^3)$ in the topology derived from the norm

$$\|\cdot\|_{\mathcal{H}} \cdot \mathcal{D}_0^4(\mathcal{R}^3) \to \mathcal{R}_+ = [0, \infty)$$

$$f = (f_0, f_1, f_2, f_3) \mapsto \|f\|_{\mathcal{H}}$$

$$= \left[\sum_{\mu=0}^{3} \int d\mathbf{x}\, |(Cf_\mu)(\mathbf{x})|^2\right]^{1/2}.$$

The Hilbert space $\mathcal{H}^*$ is the dual of the Hilbert space $\mathcal{H}$ in the pairing

$$\langle \circ, \circ \rangle: \mathcal{H}^* \times \mathcal{H} \to \mathcal{R}$$

$$(f, g) \mapsto \langle f, g \rangle = \sum_{\mu=0}^{3} \langle C^{-1}f_\mu, Cg_\mu \rangle_0,$$

where $\langle \circ, \circ \rangle_0$ is the inner product of $L^2(\mathcal{R}^3, d\mathbf{x})$.

Then, the field operators Φ_μ and π_μ, at time zero, are defined as follows:

$$\Phi_\mu(f_\mu) = [2(2\pi)^3]^{-1/2}[a_\mu^*(F_\mu^{(-)}) + a_\mu(F_\mu^{(+)})],$$

$$\pi_\mu(g_\mu) = [2(2\pi)^3]^{-1/2}[a_\mu^*(G_\mu^{(-)}) - a_\mu(G_\mu^{(+)})],$$

where

$$F_\mu^{(\pm)}(\mathbf{p}) = |\mathbf{p}|^{-1}\tilde{f}_\mu(\pm\mathbf{p}),$$

$$G_\mu^{(\pm)}(\mathbf{p}) = \tilde{g}_\mu(\pm\mathbf{p}),$$

$$f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^*, \quad g = (g_0, g_1, g_2, g_3) \in \mathcal{H}$$

and, as usual, $\tilde{h}$ denotes the Fourier transform of h.

Let $(f, g) \in \mathcal{H}^* \times \mathcal{H}$ and set

$$\sum_{\mu=0}^{3} \Phi_\mu(f_\mu) = \Phi(f), \quad \sum_{\mu=0}^{3} \pi_\mu(g_\mu) = \pi(g).$$

It is well known[14,15] that each of $\Phi(f)$ and $\pi(g)$ is essentially self-adjoint on $\mathcal{J}_{\text{alg}}$. We denote again by $\Phi(f)$ and $\pi(g)$ the closure of $\Phi(f)$ and of $\pi(g)$ respectively. The corresponding field operators for arbitrary time $t \in \mathcal{R}$ will be denoted by $\Phi(f, t)$ and $\pi(g, t)$. These latter may be written symbolically as follows:

$$\Phi(f, t) = \sum_{\mu=0}^{3} \int d\mathbf{x}\, \Phi_\mu(\mathbf{x}, t) f_\mu(\mathbf{x}),$$

$$\pi(g, t) = \sum_{\mu=0}^{3} \int d\mathbf{x}\, \pi_\mu(\mathbf{x}, t) g_\mu(\mathbf{x}),$$

where $\Phi_\mu(\cdot, \circ)$ and $\pi_\mu(\circ, \cdot)$, $\mu = 0, 1, 2, 3$, are operator-valued distributions.

Finally, we remark that there is a unitary representation U of $\mathcal{P}_+^\uparrow$, the Poincare group, on $\mathcal{J}$ such that

$$U(a, \Lambda)\, \Phi_\mu(x) U(a, \Lambda)^{-1} = \sum_{\nu=0}^{3} (\Lambda)^{-1}{}_{\mu\nu} \Phi_\nu(\Lambda x + a)$$

where $x, a \in \mathcal{R}^3 \times \mathcal{R}$, and $(a, \Lambda) \in \mathcal{P}_+^\uparrow$. The unitary operator $(a, \Lambda) \mapsto U(a, \Lambda)$ satisfies the spectrum condition in the usual sense, and it leaves fixed the vacuum vector in $\mathcal{J}$. We note also that those elements of the homogeneous subgroup of $\mathcal{P}_+^\uparrow$ corresponding to time reversal are represented on $\mathcal{J}$ by antiunitary operators.

3. THE ALGEBRA OF QUASILOCAL OBSERVABLES

Let κ denote the complex Hilbert space of four-component vector complex-valued functions z of the form $z = f + ig$, $f \in \mathcal{H}^*$, $g \in \mathcal{H}$, equipped with the inner product

$\langle\cdot,\cdot\rangle_\kappa:\kappa\times\kappa\to\mathbb{C}$

$$(z,z')\mapsto\langle z,z'\rangle_\kappa=\langle f,f'\rangle_{H^*}+\langle g,g'\rangle_H+i(\langle f',g\rangle-\langle f,g'\rangle),$$

where $z=f+ig$, $z'=f'+ig'$ and as usual, $\langle\cdot,\cdot\rangle$ denotes the bilinear pairing of H^* and H. One sees immediately that $(z,z')\to\mathrm{Im}\langle z,z'\rangle_\kappa$ is a simplectic form on κ. As substitutes for the usual canonical commutation relations for the operators $\Phi(f)$ and $\pi(g)$, $(f,g)\in H^*\times H$, there are the Weyl relations,[12,13]

$$W(z)W(z')=\exp(-\frac{i}{2}\mathrm{Im}\langle z,z'\rangle_k)W(z+z'),$$

$$(z,z')\in\kappa\times\kappa.$$

Here W is a continuous map from κ to the group $U(\mathcal{F})$ of all unitary operators on $\mathcal{F}$ such that the map $t\to W(tz)$ of R into $U(\mathcal{F})$, is strongly continuous for each fixed $z\in\kappa$. As a consequence of the last property of the map W, we have, by Stone's theorem,[16] that $W(z)=\exp[iR(z)]$, where $R(z)$ is an unbounded densely defined self-adjoint operator on $\mathcal{F}$ for each $z\in\kappa$. It is well known[17] that the operators $\Phi(f)$ and $\pi(g)$ are related to $R(z)$ as follows,

$$\Phi(f)=R(C^{-1}f),\quad \pi(g)=R(iCg),$$

where $z=f+ig\in\kappa$ with $(f,g)\in H^*\times H$: For us, the Weyl operators $\{W(z):z\in\kappa\}$ will play a significant role in the sequel.

Next, recall that the free quantum electromagnetic field comes equipped with a one-parameter group $\{\nu_t:t\in R\}$ of automorphisms which mediate the time evolution of the theory—thanks to Lorentz covariance. On $\{W(z):z\in\kappa\}$, the automorphism ν_t, $t\in R$, reduces to the following transformation,

$$\nu_t(W(z))=W((Z(t,\cdot))),$$

where

$$Z(t,\mathbf{x})=F(t,\mathbf{x})+iG(t,\mathbf{x})$$

and

$$(t,\mathbf{x})\mapsto F(t,\mathbf{x}),\ (t,\mathbf{x})\mapsto G(t,\mathbf{x})$$

are the unique solutions of the equations

$$\frac{\partial F}{\partial t}=G,\quad \frac{\partial G}{\partial t}=\Delta F$$

with Cauchy data $F(0,\mathbf{x})=f(\mathbf{x})$, $G(0,\mathbf{x})=g(\mathbf{x})$, $(f,g)\in H^*\times H$.

The algebra $\mathfrak{A}$ of quasilocal observables can now be introduced. To this end, let $C=\{O_i\}_{i=1}^\infty$ be a covering of M^4 with closed double cones (in M^4). Then for $O\in C$, let $\mathfrak{A}(O)$ denote the W^*-algebra generated by $\{W(Z(\cdot,\cdot)):\mathrm{supp}\,Z(\cdot,\cdot)\subset O\}$. The assignment $O\to\mathfrak{A}(O)$ has the following properties:

(i) If O_1, $O_2\in C$ with $O_1\subset O_2$, then $\mathfrak{A}(O_2)\supset\mathfrak{A}(O_1)$;

(ii) If O_1, $O_2\in C$ are mutually spacelike separated, then $[\mathfrak{A}(O_1),\mathfrak{A}(O_2)]=0$, where $[A,B]=AB-BA$. Then $\mathfrak{A}$ is the C^*-algebra defined as the closure in the uniform topology of $\cup_{O\in C}\mathfrak{A}(O)$, i.e., $\mathfrak{A}=\overline{\cup_{O\in C}\mathfrak{A}(O)}$.

(iii) The Poincare group $P_+^\uparrow$ is represented by automorphisms τ_L of $\mathfrak{A}$, $L\in P_+^\uparrow$, and we have $\tau_L(\mathfrak{A}(O))=\mathfrak{A}(LO)$ where $LO=\{x=(x_0,\mathbf{x})\in M^4:L^{-1}x\in O\}$.

(iv) As mentioned in Sec. 2, Fock space $\mathcal{F}$ carries a strongly continuous unitary representation U of $P_+^\uparrow$ which implements τ_L, $L\in P_+^\uparrow$; furthermore, U satisfies the spectrum condition.

(v) $\mathfrak{A}$ acts irreducibly on $\mathcal{F}$.

Remark: In the next section, by considering $\mathfrak{A}$ as an abstract C^*-algebra, we construct some inequivalent irreducible $*$-representations of $\mathfrak{A}$ which possess most of the properties (i)—(v) listed above.

4. $*$-REPRESENTATIONS OF $\mathfrak{A}$

In this section, we consider a wide class of $*$-automorphisms of $\mathfrak{A}$ which lead to inequivalent irreducible $*$-representations of $\mathfrak{A}$, regarded as an abstract C^*-algebra. But first, we begin with some notation.

Let $\{\varphi_n=(\varphi_{n\mu}:\mu=0,1,2,3)\}_{n=1}^\infty$ be a complete orthogonal set from H^*. Then, given any set $\{a_n=(a_{n\mu}:\mu=0,1,2,3)\}_{n=1}^\infty$ of infinite sequences of nonvanishing real numbers, we can find, by the Riesz representation theorem, a dual complete orthogonal set $\{\xi_n=(\xi_{n\mu}:\mu=0,1,2,3)\}_{n=1}^\infty$ from H such that

$$\langle\varphi_{n\mu},\xi_{m\nu}\rangle_0=\delta_{nm}\delta_{\mu\nu}a_{n\mu}=\delta_{nm}\delta_{\mu\nu}a_{m\nu}$$

where, as before $\langle\cdot,\cdot\rangle_0$ denotes the inner product of $L^2(R^3,d\mathbf{x})$.

For $O\in C$, the class of all closed double cones in M^4, let φ_n^O and ξ_n^O be defined on O as follows:

$$\varphi_n^O(x_0,\mathbf{x})=\varphi_n(\mathbf{x}),\quad (x_0,\mathbf{x})\in O,$$

$$\xi_n^O(x_0,\mathbf{x})=\xi_n(\mathbf{x}),\quad (x_0,\mathbf{x})\in O.$$

Let $\{c_n^O=(c_{n\mu}^O:\mu=0,1,2,3)\}_{n=1}^\infty$ and $\{d_n^O=(d_{n\mu}^O:\mu=0,1,2,3)\}_{n=1}^\infty$ be two sets of infinite sequences of nonvanishing real numbers, depending on $O\in C$, such that

$$\lim_{O\to M^4}c_{n\mu}^O=c_{n\mu},\quad \lim_{O\to M^4}d_{n\mu}^O=d_{n\mu},$$

$$n=1,2,\ldots,\infty,\quad \mu=0,1,2,3,\ \text{exist.}$$

In the sequel, we are interested only in the class $\mathfrak{S}(H^*,H)$ of sequences $\{(\{c_n^O\}_{n=1}^\infty,\{d_n^O\}_{n=1}^\infty):O\in C\}$ such that the following two assumptions are satisfied:

Assumption (A)

$$\sum_{n=1}^\infty\{\|c_n^O\cdot\varphi_n^O\|_{H^*}^2+\|d_n^O\cdot\xi_n^O\|_H^2\}<\infty$$

where $c_n^O\cdot\varphi_n^O=(c_{n\mu}^O\varphi_{n\mu}^O:\mu=0,1,2,3)$ and similarly for $d_n^O\cdot\xi_n^O$, $O\in C$. This assumption is readily satisfied. See Ref. 18 and also below, for example. As emerges below, this assumption ensures the essential self-adjointness of a certain operator sum with an infinite number of operator summands.

Assumption (B)

$$\sum_{n=1}^\infty s_{n\mu}=\infty,\quad \mu=0,1,2,3$$

where $s_{n\mu}=\sin^4\theta_{n\mu}$ and $\theta_{n\mu}=c_{n\mu}d_{n\mu}a_{n\mu}$.

Remark: It is perhaps instructive to exhibit a choice of the sets of real numbers $\{c_n^O\}_{n=1}^\infty, \{d_n^O\}_{n=1}^\infty$, and $\{a_n\}_{n=1}$ which belong to $\mathfrak{S}(\mathcal{H}^*, \mathcal{H})$. To this end, let J denote the set of all infinite sequences $\{\lambda_n\}_{n=1}^\infty$ such that $\max_n|\lambda_n|<\infty$. $\min_n|\lambda_n|<\infty$. It is clear that J is in fact a linear space under the usual notions of addition and scalar multiplication. For $\{c_{n\mu}\}_{n=1}^\infty$ and $\{d_{n\mu}\}_{n=1}^\infty$, $\mu=0, 1, 2, 3$ belonging to J, set

$$c_{n\mu}^O = c_{n\mu}\exp\left(-\frac{\beta n^2}{1+|O|^2}\right),\ c_{n\mu}\neq 0,\quad \beta>0,$$

$$d_{n\mu}^O = d_{n\mu}\exp\left(-\frac{\alpha n^2}{1+|O|^2}\right),\quad d_{n\mu}\neq 0,\ \alpha>0,$$

$$a_{n\mu}=\frac{(-1)^n\pi}{2c_{n\mu}\,d_{n\mu}},$$

where $|O|$ denotes the Lebesgue measure of $O\in\mathcal{C}$. Clearly, $\lim_{O\to M^4} c_{n\mu}^O = c_{n\mu}$, $\lim_{O\to M^4} d_{n\mu}^O = d_{n\mu}$ and it is easy to check that

$$\sum_{n=1}^\infty \{\|c_n^O\circ\varphi_n^O\|_{\mathcal{H}^*}^2 + \|d_n^O\circ\xi_n^O\|_{\mathcal{H}}^2\}<\infty,$$

hence Assumption (A) is satisfied. Assumption (B) is also satisfied, for

$$\sin c_{n\mu}\,d_{n\mu}\,a_{n\mu}=\sin(-1)^n\frac{\pi}{2}=(-1)^n\sin\frac{\pi}{2}=(-1)^n.$$

Hence

$$s_{n\mu}=(-1)^{4n}=1\quad\text{and}\quad\sum_{n=1}^\infty s_{n\mu}=\sum_{n=1}^\infty 1=\infty,\quad \mu=0, 1, 2, 3.$$

Remark: We shall now introduce certain linear operators on, and between, the spaces $\mathcal{H}^*$ and $\mathcal{H}$. We need the operators in connection with the construction of a class of $*$-automorphisms of $\mathfrak{A}$ which we define below. Thus, let

$$\mathcal{A}:\mathcal{H}^*\to\mathcal{H},\quad \mathcal{B}:\mathcal{H}^*\to\mathcal{H},\ \mathcal{T}:\mathcal{H}\to\mathcal{H},\ \mathcal{L}:\mathcal{H}\to\mathcal{H}^*$$

be defined as follows:

$$(\mathcal{A}f)_\mu=\sum_{n=1}^\infty\frac{\langle f_\mu,\xi_{n\mu}\rangle_0(\cos 2\theta_{n\mu}-1)}{a_{n\mu}}\circ\varphi_{n\mu}$$
$$\equiv\mathcal{A}_\mu f_\mu,\quad f\in\mathcal{H}^*,\quad \mu=0, 1, 2, 3,$$

$$(\mathcal{B}f)_\mu=\sum_{n=1}^\infty\frac{\langle f_\mu,\xi_{n\mu}\rangle_0\sin 2\theta_{n\mu}}{a_{n\mu}}\cdot\xi_{n\mu}$$
$$\equiv\mathcal{B}_\mu f_\mu,\ f\in\mathcal{H}^*,\quad \mu=0, 1, 2, 3,$$

$$(\mathcal{T}g)_\mu=\sum_{n=1}^\infty\frac{\langle\varphi_{n\mu},g_\mu\rangle_0(\cos 2\theta_{n\mu}-1)}{a_{n\mu}}\cdot\xi_{n\mu}$$
$$\equiv\mathcal{T}_\mu g_\mu,\quad g\in\mathcal{H},\quad \mu=0, 1, 2, 3,$$

$$(\mathcal{L}g)_\mu=\sum_{n=1}^\infty\frac{\langle\varphi_{n\mu},g_\mu\rangle_0\sin 2\theta_{n\mu}}{a_{n\mu}}\circ\varphi_{n\mu}$$
$$\equiv\mathcal{L}_\mu g_\mu,\quad g\in\mathcal{H},\quad \mu=0, 1, 2, 3.$$

The operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{L}$ are bounded and we have the following trivially verifiable estimates

$$\|\mathcal{A}f\|_{\mathcal{H}^*}^2\leq 2\|f\|_{\mathcal{H}^*}^2,\quad f\in\mathcal{H}^*,$$

$$\|\mathcal{B}f\|_{\mathcal{H}}^2\leq 2b\|f\|_{\mathcal{H}^*}^2,\quad f\in\mathcal{H}^*,\quad b>0,$$

$$\|\mathcal{T}g\|_{\mathcal{H}}^2\leq 2\|g\|_{\mathcal{H}}^2,\quad g\in\mathcal{H},$$

$$\|\mathcal{L}g\|_{\mathcal{H}^*}^2\leq 2e\|g\|_{\mathcal{H}}^2,\quad g\in\mathcal{H},\quad e>0.$$

Denote again by $\mathfrak{S}(\mathcal{H}^*, \mathcal{H})$ the class of all operators $(\mathcal{A}, \mathcal{B}, \mathcal{T}, \mathcal{L})$ obtained by employing different choices of $(\{c_n^O\}_{n=1}^\infty, \{d_n^O\}_{n=1}^\infty\}: O\in\mathcal{C})\in\mathfrak{S}(\mathcal{H}^*, \mathcal{H})$ in the definition, as above, of these operators. In the sequel, we utilize the following assertion:

Theorem 1: The bounded linear operator

$$Y_{11}: L^2(\mathcal{R}^3, d\mathbf{x})\to L^2(\mathcal{R}^3, d\mathbf{x})$$

given by

$$Y_{11}=C^{-1}\mathcal{A}_1C+C\mathcal{A}_1^tC^{-1}+C\mathcal{A}_1^tC^{-2}\mathcal{A}_1C+C^{-1}\mathcal{L}_1^t\mathcal{B}_1C,$$

$(\mathcal{A}, \mathcal{B}, \mathcal{T}, \mathcal{L})\in\mathfrak{S}(\mathcal{H}^*, \mathcal{H})$, is not of the Hilbert–Schmidt class. [Here A^t denotes the transpose of A and $C=(-\Delta)^{1/4}$.]

Proof: The bounded linear transformation Y_{11} is an integral operator which acts as follows in $L^2(\mathcal{R}^3, d\mathbf{x})$,

$$Y_{11}h=\sum_{n=1}^\infty\Big(\frac{(\cos 2\theta_{n1}-1)}{a_{n1}}[\langle h, C\xi_{n1}\rangle_0 C^{-1}\varphi_{n1}$$
$$+\langle h, C^{-1}\varphi_{n1}\rangle_0C\xi_{n1}+\frac{(\cos 2\theta_{n1}-1}{a_{n1}}\|\varphi_{n1}\|_{-1}^2$$
$$\times\langle h, C\xi_{n1}\rangle_0C\xi_{n1}]\Big\}$$
$$+(-1)\sum_{n=1}^\infty\frac{\langle h, C\xi_{n1}\rangle_0\sin^2 2\theta_{n1}}{a_{n1}}C^{-1}\varphi_{n1},$$

where $h\in L^2(\mathcal{R}^3, d\mathbf{x})$ and $\|\eta\|_k=\|C^k\eta\|_{L^2(\mathcal{R}^3, d\mathbf{x})}$, $k\in\mathcal{R}$. Hence, Y_{11} has a kernel whose Fourier transform $\tilde{Y}_{11}(\cdot,\cdot)$ is

$$\tilde{Y}_{11}(\mathbf{p},\mathbf{q})=\sum_{n=1}^\infty\tilde{Y}_{11,n}(\mathbf{p},\mathbf{q})$$

where

$$\tilde{Y}_{11,n}(\mathbf{p},\mathbf{q})$$
$$=\frac{(\cos 2\theta_n-1)}{a_{n1}}|\mathbf{p}|^{-1/2}\tilde{\varphi}_{n1}(\mathbf{p})|\mathbf{q}|^{1/2}\tilde{\xi}_{n1}(\mathbf{q})$$
$$+\frac{(\cos 2\theta_{n_1}-1)}{a_{n1}}|\mathbf{p}|^{1/2}\tilde{\xi}_{n1}(\mathbf{p})|\mathbf{q}|^{-1/2}\varphi_{n1}(\mathbf{q})$$
$$+\frac{(\cos 2\theta_{n_1}-1)^2}{a_{n1}^2}\|\varphi_{n1}\|_{-1}^2|\mathbf{p}|^{1/2}\tilde{\xi}_{n1}(\mathbf{p})|\mathbf{q}|^{1/2}\tilde{\xi}_{n1}(\mathbf{q})$$
$$-\frac{\sin^2 2\theta_{n_1}}{a_{n1}}|\mathbf{p}|^{-1/2}\tilde{\varphi}_{n1}(\mathbf{p})|\mathbf{q}|^{1/2}\tilde{\xi}_{n1}(\mathbf{q}).$$

Hence

$$\int d\mathbf{p}\,d\mathbf{q}\,|\tilde{Y}_{11,n}(\mathbf{p},\mathbf{q})|^2$$
$$=\frac{1}{a_{n1}^2}[(2\sin^2\theta_{n1}+4\sin^2\theta_{n1}\cos^2\theta_{n1})^2+12\sin^4\theta_{n1}$$
$$-8(2\sin^2\theta_{n1}+4\sin^2\theta_{n1}\cos^2\theta_{n1})\sin^4\theta_{n1}]\|\varphi_{n1}\|_{-1}^2\|\xi_{n1}\|_{+1}^2$$
$$+\frac{16\sin^2\theta_{n1}}{a_{n1}^4}\|\varphi_{n1}\|_{-1}^4\|\xi_{n1}\|_{+1}^4+4(2\sin^2\theta_{n1}$$
$$+4\sin^2\theta_{n1}\cos^2\theta_{n1})\sin^2\theta_{n1}$$
$$=\frac{48\sin^4\theta_{n1}\cos^4\theta_{n1}}{a_{n1}^2}\|\varphi_{n1}\|_{-1}^2\|\xi_{n1}\|_{+1}^2+\frac{16\sin^8\theta_{n1}}{a_{n1}^4}\|\varphi_{n1}\|_{-1}^4$$
$$\times\|\xi_{n1}\|_{+1}^4+8\sin^4\theta_{n1}+16\sin^4\theta_{n1}\cos^2\theta_{n1}.$$

Next, since $a_{n1}=\langle\varphi_{n1},\xi_{n1}\rangle_0$, it follows that

$$a_{n1}^2 \leqslant \|\varphi_{n1}\|_{-1}^2 \, \|\xi_{n1}\|_{+1}^2.$$

Hence

$$\int d\mathbf{p}\, d\mathbf{q}\, |\tilde{Y}_{11,n}(\mathbf{p},\mathbf{q})|^2 \geqslant$$

$$\geqslant 48\sin^4\theta_{n1}\cos^4\theta_{n1} + \sin^8\theta_{n1} + 8\sin^4\theta_{n1} + 16\sin^4\theta_{n1}\cos^2\theta_{n1}$$

$$\geqslant 8\sin^4\theta_{n1}.$$

Thus

$$\int d\mathbf{p}\, d\mathbf{q}\, |\tilde{Y}_{11}(\mathbf{p},\mathbf{q})|^2 = \sum_{n=1}^{\infty}\int d\mathbf{p}\, d\mathbf{q} |\tilde{Y}_{11,n}(\mathbf{p},\mathbf{q})|^2 \geqslant 8\sum_{n=1}^{\infty}\sin^4\theta_{n1} = \infty,$$

by Assumption (B). Hence, the operator Y_{11} is not of Hilbert–Schmidt class. This concludes the proof.

Remark: We are now in a position to discuss a certain class of $*$-automorphisms of $\mathfrak{A}$ constructed by means of the operators $(\mathcal{A},\mathcal{B},\mathcal{T},\mathcal{L}) \in \mathfrak{S}(\mathcal{H}^*,\mathcal{H})$. We begin with the following assertion:

Theorem 2: The transformation

$$\gamma: \begin{pmatrix}\Phi(f)\\ \pi(g)\end{pmatrix} \to \begin{pmatrix}\Phi_\gamma(f)\\ \pi_\gamma(g)\end{pmatrix} = \begin{pmatrix}\Phi((I+\mathcal{A})f) + \pi(\mathcal{B}f)\\ \pi((I+\mathcal{T})g) + \Phi(\mathcal{L}g)\end{pmatrix},$$

$(f,g) \in \mathcal{H}^*\times\mathcal{H}$, $(\mathcal{A},\mathcal{B},\mathcal{T},\mathcal{L}) \in (\mathcal{H}^*,\mathcal{H})$, preserves the canonical commutation relations between Φ and π.

Proof: The assertion is readily verified by formal manipulations of the canonical commutation relations between the field operators Φ and π.

Remark: It is well known that an automorphism is uniquely determined on all of $\mathfrak{A}$ if its action on the time zero algebra $\mathfrak{A}_0$, which is the C^*-algebra of quasilocal observables generated by $\{W(z): z\in\kappa\}$, is known. The transformation γ of the last theorem induces and is induced by a $*$-automorphism γ_T of $\mathfrak{A}$ described in the next theorem. The $*$-automorphisms γ_T lead to inequivalent irreducible $*$-representations of $\mathfrak{A}$.

Theorem 3: The transformation γ indicated in the last theorem is not unitarily implementable.

Proof: As remarked above, the transformation γ is the induced action of a certain $*$-automorphism γ_T which acts as follows,

$$\gamma_T: W(z) \to W(Tz)$$

on elements of $\mathfrak{A}$ of the form $W(z)$, $z\in\kappa$, where T is some simplectic transformation.[12] Next, we compute γ_T and hence also T.

Since $\Phi(f) = R(C^{-1}f)$, $\pi(g) = R(iCg)$, $(f,g)\in\mathcal{H}^*\times\mathcal{H}$, then under the action of γ,

$$\Phi(f) = R(C^{-1}f) \mapsto \Phi_\gamma(f) = R(C^{-1}(I+\mathcal{A})f) + R(iC\mathcal{B}f)$$

or

$$R(f) \mapsto R(C^{-1}(I+\mathcal{A})Cf) + R(iC\mathcal{B}\,Cf).$$

Similarly,

$$\pi(g) = R(iCg) \mapsto \pi_\gamma(g) = R(iC(I+\mathcal{T})g) + R(C^{-1}\mathcal{L}g)$$

or

$$R(ig) \mapsto R(iC(I+\mathcal{T})C^{-1}g) + R(C^{-1}\mathcal{L}\,C^{-1}g).$$

Thus, there is the following transformation

$$R(f+ig) \mapsto R(C^{-1}(I+\mathcal{A})Cf + C^{-1}\mathcal{L}C^{-1}g + iC\mathcal{B}\,Cf + i\,C(I+\mathcal{T})C^{-1}g).$$

Hence, we can identify the mappings:

$$f \mapsto C^{-1}(I+\mathcal{A})Cf + C^{-1}\mathcal{L}C^{-1}g,$$

$$g \mapsto C\mathcal{B}\,Cf + C(I+\mathcal{T})C^{-1}g.$$

These may be viewed concisely as a transformation of $\mathcal{H}^*\oplus\mathcal{H}$ onto itself, as follows:

$$T: \begin{bmatrix} f_0\\ \vdots\\ f_3\\ g_0\\ \vdots\\ g_3\end{bmatrix} \to T\begin{bmatrix} f_0\\ \vdots\\ f_3\\ g_0\\ \vdots\\ g_3\end{bmatrix},$$

$f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^*$, $\quad g = (g_0, g_1, g_2, g_3) \in \mathcal{H}$, where T is the following matrix whose entries are operators:

$$T = \begin{bmatrix} M_1 & 0 & 0 & 0 & Q_1 & 0 & 0 & 0\\ 0 & M_2 & 0 & 0 & 0 & Q_2 & 0 & 0\\ 0 & 0 & M_3 & 0 & 0 & 0 & Q_3 & 0\\ 0 & 0 & 0 & M_4 & 0 & 0 & 0 & Q_4\\ P_1 & 0 & 0 & 0 & N_1 & 0 & 0 & 0\\ 0 & P_2 & 0 & 0 & 0 & N_2 & 0 & 0\\ 0 & 0 & P_3 & 0 & 0 & 0 & N_3 & 0\\ 0 & 0 & 0 & P_4 & 0 & 0 & 0 & N_4 \end{bmatrix}$$

and

$$M_\mu = C^{-1}(I+\mathcal{A}_\mu)C, \quad N_\mu = C(I+\mathcal{T}_\mu)C^{-1},$$

$$P_\mu = C\mathcal{B}_\mu C, \quad Q_\mu = C^{-1}\mathcal{L}_\mu C^{-1}, \quad \mu = 0, 1, 2, 3.$$

Let T^t denote the matrix of operators obtained from T by replacing each entry of T with its transpose. Then, by Shale's theorem,[19] γ_T, and hence γ, is unitarily implementable if and only if $Y = T^tT - I$ is of Hilbert–Schmidt class, in the sense that each entry of Y is of Hilbert–Schmidt class. Writing $Y = (Y_{ij})$, where Y_{ij} is an operator for $i, j = 0, 1, \ldots, 7$, we have, for example, that

$$Y_{11} = C^{-1}\mathcal{A}_1 C + C\mathcal{A}_1^t C^{-1} + C\mathcal{A}_1^t C^{-2}\mathcal{A}_1 C + C^{-1}\mathcal{L}_1^t\mathcal{B}_1 C.$$

By theorem 1, Y_{11} is not of Hilbert–Schmidt class. Thus the entries of the matrix Y do not all consist of Hilbert–Schmidt operators. Hence, γ is not unitarily implementable. This completes the proof.

Remark: It is useful to know whether or not the automorphism γ_T, or equivalently γ, is locally unitarily implementable. To this end, we have

Theorem 4: The transformation γ, or equivalently γ_T, is locally unitarily implementable.

Proof: We shall prove the claim by exhibiting a unitary operator which implements γ locally.

Let $\mathcal{O} \in \mathcal{C}$ and set

$$\sum_{n=1}^{m} \sum_{\mu=0}^{3} [(c_{n\mu}^{\mathcal{O}})^2 (\Phi_\mu(\varphi_{n\mu}^{\mathcal{O}}))^2 + (d_{n\mu}^{\mathcal{O}})^2 (\pi_\mu(\xi_{n\mu}^{\mathcal{O}}))^2]$$

$$= Q_m(\mathcal{O}).$$

Then for each m, $Q_m(\mathcal{O})$ is a symmetric operator which is essentially self-adjoint on a dense invariant domain in $\mathcal{J}$. The strong limit $Q(\mathcal{O})$ of $Q_m(\mathcal{O})$ as m tends to infinity exists, is symmetric and is also essentially self-adjoint on a dense invariant domain in $\mathcal{J}$, thanks to Assumption (A). We denote again by $Q(\mathcal{O})$ the closure of $Q(\mathcal{O})$.

Let $\Gamma(\mathcal{O})$, $\mathcal{O} \in \mathcal{C}$, be the unitary operator

$$\Gamma(\mathcal{O}) = \exp[iQ(\mathcal{O})].$$

Let $\mathcal{A}^{\mathcal{O}}$, $\mathcal{B}^{\mathcal{O}}$, $\mathcal{T}^{\mathcal{O}}$, $\mathcal{L}^{\mathcal{O}}$, be the operators defined in the same way as the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$, $\mathcal{L}$ but using the sequences $\{\varphi_n^{\mathcal{O}} = (\varphi_{n\mu}^{\mathcal{O}},\ \mu = 0, 1, 2, 3)\}_{n=1}^{\infty}$ and $\{\xi_n^{\mathcal{O}} = (\xi_{n\mu}^{\mathcal{O}} : \mu = 0, 1, 2, 3)\}_{n=1}^{\infty}$ which belong to $\mathcal{H}^*$ and $\mathcal{H}$ respectively and the numbers $\{c_n^{\mathcal{O}} = (c_{n\mu}^{\mathcal{O}} : \mu = 0, 1, 2, 3)\}$ and $\{d_n^{\mathcal{O}} = (d_{n\mu}^{\mathcal{O}} : \mu = 0, 1, 2, 3)\}$. Define f and g on $\mathcal{O} \in \mathcal{C}$ as follows:

$$\{f^{\mathcal{O}}(x_0, \mathbf{x}) = f(\mathbf{x}), \quad (x_0, \mathbf{x}) \in \mathcal{O},$$

$$\{g^{\mathcal{O}}(x_0, \mathbf{x}) = g(\mathbf{x}), \quad (x_0, \mathbf{x}) \in \mathcal{O},$$

with $(f, g) \in \mathcal{H}^* \times \mathcal{H}$. Then the following results may be readily checked by means of the canonical commutation relations between the operators Φ and π, and by a careful use of the Baker–Campbell–Hausdorff formula:

$$\Phi((I + \mathcal{A}^{\mathcal{O}})f^{\mathcal{O}}) + \pi(\mathcal{B}^{\mathcal{O}} f^{\mathcal{O}}) = \Phi_{\gamma(\mathcal{O})}(f^{\mathcal{O}})$$
$$= \Gamma(\mathcal{O})\Phi(f^{\mathcal{O}})\Gamma(\mathcal{O})^{-1},$$

$$\pi((I + \mathcal{T}^{\mathcal{O}})g^{\mathcal{O}}) + \Phi(\mathcal{L}^{\mathcal{O}} g^{\mathcal{O}}) = \pi_{\gamma(\mathcal{O})}(g^{\mathcal{O}})$$
$$= \Gamma(\mathcal{O})\pi(g^{\mathcal{O}})\Gamma(\mathcal{O})^{-1}.$$

This concludes the proof.

Remark: The preceding theorem has the interpretation that the $*$-automorphism γ_T of $\mathfrak{A}$ is locally unitarily implementable. Indeed, the local transformation

$$\gamma(\mathcal{O}) : \begin{pmatrix} \Phi(f^{\mathcal{O}}) \\ \pi(g^{\mathcal{O}}) \end{pmatrix} \rightarrow \begin{pmatrix} \Phi_{\gamma(\mathcal{O})}(f^{\mathcal{O}}) \\ \pi_{\gamma(\mathcal{O})}(g^{\mathcal{O}}) \end{pmatrix},$$

with the operators $\Phi_{\gamma(\mathcal{O})}(f^{\mathcal{O}})$ and $\pi_{\gamma(\mathcal{O})}(g^{\mathcal{O}})$ as indicated in the last theorem, induces and is induced by a local $*$-automorphism $\gamma_T(\mathcal{O})$, $\mathcal{O} \in \mathcal{C}$ and one has that $\gamma_T(\mathcal{O}_2)$ extends $\gamma_T(\mathcal{O}_1)$ if $\mathcal{O}_2 \supset \mathcal{O}_1$.

Hence γ_T is the inductive limit of the net $\{\gamma_T(\mathcal{O}) : \mathcal{O} \in \mathcal{B}\}$ of local $*$-automorphisms of $\mathfrak{A}$.

Definition: An automorphism ρ of $\mathcal{O}$ is said to be localized[4] in $\mathcal{O} \in \mathcal{C}$ if the restriction of ρ to $\mathfrak{A}(\mathcal{O}')$ is the identity automorphism, where $\mathcal{O}'$ is here, and hereafter, the causal complement of $\mathcal{O} \in \mathcal{C}$.

Theorem 5: The $*$-automorphism $\gamma_T(\mathcal{O})$ of $\mathcal{O}$ is not localized in $\mathcal{O} \in \mathcal{C}$.

Proof: Let $T(\mathcal{O})$ be the simplectic transformation determined by $(\mathcal{A}^{\mathcal{O}}, \mathcal{B}^{\mathcal{O}}, \mathcal{T}^{\mathcal{O}}, \mathcal{L}^{\mathcal{O}})$, $\mathcal{O} \in \mathcal{C}$. Then for $z^{\mathcal{O}'} = f^{\mathcal{O}'} + ig^{\mathcal{O}'} \in \kappa$, we have

$$\gamma_T(\mathcal{O})(W(z^{\mathcal{O}'})) = W(T(\mathcal{O})z^{\mathcal{O}'}).$$

But $T(\mathcal{O})$, $\mathcal{O} \in \mathcal{C}$, is pseudolocal[17] and it is readily checked from the explicit expression [which involves the pseudodifferential operators $(-\Delta)^{\pm 1/4}$] for $T(\mathcal{O})$ that $T(\mathcal{O})$ is not the identity operator on the subspace of κ spanned by vector functions of the form $z\mathcal{O}' = f\mathcal{O}' + ig^{\mathcal{O}'}$. Hence $\gamma_T(\mathcal{O}) \mid_{\mathfrak{A}(\mathcal{O}')}$ is not the identity automorphism. Hence $\gamma_T(\mathfrak{A})$ is not localized in $\mathcal{O} \in \mathcal{C}$. This completes the proof.

Remark: The preceding result is as one would expect, for, here, we are dealing with a theory which exhibits long range forces and therefore automorphisms of $\mathfrak{A}$ cannot be localized to bounded regions of M^4. This is in contradistinction to the situation in strong interaction theories, where short range forces operate and there are localized automorphisms. We refer to the Introduction of Ref. 5 for further comments on this subject.

Remark: We recall that each simplectic transformation T is determined by a quadruple $(\mathcal{A}, \mathcal{B}, \mathcal{T}, \mathcal{L}) \in \mathfrak{S}(\mathcal{H}^*, \mathcal{H})$. We denote now by $\mathfrak{S}_0(\mathcal{H}^*, \mathcal{H})$ the collection of all simplectic transformations obtainable by making different choices of the quadrupule $(\mathcal{A}, \mathcal{B}, \mathcal{T}, \mathcal{L})$ from $\mathfrak{S}(\mathcal{H}^*, \mathcal{H})$.

Next, regard $\mathfrak{A}$ now as an abstract C^*-algebra and let ρ_0 be the representation of $\mathfrak{A}$ by itself. We obtain new $*$-representations ρ_T of $\mathfrak{A}$ by forming the composition

$$\rho_T = \rho_0 \circ \gamma_T, \quad T \in \mathfrak{S}_0(\mathcal{H}^*, \mathcal{H}).$$

The new $*$-representations $\{\rho_T : T \in \mathfrak{S}_0(\mathcal{H}^*, \mathcal{H})\}$ satisfy all the axioms in Ref. 2, except possibly Poincare covariance. In Ref. 7, we study the question of Poincare covariance of the $*$-representations $\{\rho_T : T \in \mathfrak{S}_0(\mathcal{H}^*, \mathcal{H})\}$. If ρ_T, $T \in \mathfrak{S}_0(\mathcal{H}^*, \mathcal{H})$, satisfies the spectrum condition, i.e., if space–time automorphisms are implemented in ρ_T by a strongly continuous unitary representation (of M^4) whose infinitesimal generator has spectrum contained in $\{p \in (M^4)^* : [p, p] \geq 0,\ p_0 \geq 0\}$, then ρ_T is said to be positive. By Shale's theorem, we have that for $T_1, T_2 \in \mathfrak{S}_0(\mathcal{H}^*, \mathcal{H})$, the $*$-representations ρ_{T_1} and ρ_{T_2} of $\mathfrak{A}$ are unitarily implementable if and only if $Y^{(2)} - Y^{(1)}$ is Hilbert–Schmidt, where $Y^{(i)} = T_i^* T_i - I$. By a superselection sector, we mean a unitary equivalence class $\hat{\rho}_T$ of positive $*$-representations of $\mathfrak{A}$. As remarked in the Introduction, we defer to Ref. 7 the complete characterization of the subclass of the class of $*$-representations constructed above which consists of positive $*$-representations of $\mathfrak{A}$. In our analysis, in this regard, we employ certain results of Refs. 8 and 20.

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The optical group and its subgroups[a)]

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The optical group Opt(3,1) is a ten-dimensional maximal subgroup of the conformal group of space–time, characterized by the fact that it leaves a lightlike vector subspace in Minkowski space invariant. Thus it is the group underlying the symmetry structure of the parton model in particle physics. The present article is devoted to a complete classification of all closed connected subgroups of Opt(3,1). A list of representatives of all Lie subalgebras of the algebra opt(3,1) is given in the form of tables and many of their properties are established (their invariants, normalizers, isomorphism classes, etc.). Most of the subalgebras of opt(3,1) are also contained in the similitude algebra sim(3,1). We discuss a method for extracting the "new" subalgebras of opt(3,1) from the list; these will go over into a future list of subalgebras of the conformal Lie algrebra itself.

1. INTRODUCTION

The purpose of this article is to provide a complete classification of all closed continuous subgroups of a certain physically interesting ten-dimensional Lie group which we shall call the "optical group" Opt(3,1).

The group Opt(3,1) can be characterized as being the maximal subgroup of the conformal group of space—time leaving a lightlike one-dimensional vector subspace of Minkowski space invariant. According to our opinion this group is of considerable physical and mathematical interest and has so far not received the attention it deserves. Indeed, this group should make its appearance, at least implicitly, in any physical theory in which conformal invariance and lightlike particle states have a role to play. Much of the simplicity and attractiveness of the "parton" model in high energy physics[1–3] is related to the fact that in an infinite momentum frame[4] hadrons appear as static collections of partons and many aspects of their kinematics become nonrelativistic. From the group theoretical point of view this is related to the fact that an eight-dimensional subgroup of the Poincaré group (generated by L_3, K_3, L_2+K_1, L_1-K_2, P_0, P_1, P_2, and P_3, where L_i generate rotations, K_i Lorentz boosts, and P_μ translations) is associated with the lightlike vector determining the infinite momentum frame. This group in turn contains a subgroup isomorphic to the Galilei group in two space dimensions (generated by L_3, L_2+K_1, L_1-K_2, P_1, P_2, and P_0-P_3), and this provides the nonrelativistic kinematics (in the "transverse" plane).[3] If the conformal group of space time is taken as the fundamental kinematic group instead of the Poincaré group, then the above eight-dimensional group is enlarged to the optical group Opt(3,1). The Galilei group will be replaced by a nine-dimensional group, namely the so-called extended Schrödinger group Sch_2, leaving the time-dependent free Schrödinger equation in two dimensions invariant. In a previous publication[5] we have classified all continuous subgroups of Sch_2 into conjugacy classes and also gave references to earlier work (e.g., Refs. 6 and 7) on the Schrödinger group.

In a relativistic context Opt(3,1) as a subgroup of the conformal group has been used by Domokos[8] when constructing lightlike particle states. These were used in the formulation of a field theory in terms of "lightlike" components, the aim of which is to provide a theoretical basis for the quark—parton model. Implicitly the group Opt(3,1) has been used in attempts to unify dual models with light cone physics[9] [explicitly, Del Giudice *et al.* use the five-dimensional "homogeneous" factor group $\mathrm{Opt}(3,1)/W_2$, where W_2 is the five-dimensional Weyl group]. Explicitly, Opt(3,1) and its nonrelativistic implications in the context of conformal invariance and "Schrödinger invariance" were studied elsewhere.[10,11] This group can also be used to introduce a consistent definition of a position operator for a massless particle.[12]

From the mathematical point of view, it is interesting to note that Opt(3,1) contains all maximal solvable subgroups of SU(2,2) except the compact Cartan subgroup.[13] It has the structure of a semidirect product with a five-dimensional non-Abelian invariant subgroup. A classification of its closed continuous subgroups is hence a quite nontrivial application of a previously proposed classification algorithm.[14]

This article can be considered to be part of a general program, the aim of which is to provide a complete classification of all subgroups of all groups of interest in physics. On a more modest level, it completes the classification of the Lie subgroups of the most interesting maximal subgroups of the conformal group of space time (those of the Poincaré group,[14,15] similitude group,[14,16] de Sitter groups,[17,18] and extended Schrodinger group[5] have already been obtained).

The motivation for a study of the subgroup structure of a given Lie group has been discussed earlier.[13–18] We would only like to reiterate its importance for a systematic study of symmetry breaking,[19,20] its relation to the construction of group representations and bases for group representations,[21,22] and specially its relation to special function theory.[21–26]

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2. THE OPTICAL GROUP

A. The group Opt(3,1) as a subgroup of the conformal group

Let us consider the conformal group C(3,1) of (compactified) Minkowski space. It is generated by infinitesimal rotations L_i, proper Lorentz transformations K_i, translations P_μ, the dilation D, and proper conformal transformations C_μ ($i=1, 2, 3$, $\mu=0, 1, 2, 3$). These operators satisfy the commutation relations

$$\begin{array}{lll} [L_i, L_k]=\epsilon_{ikl}L_l, & [L_i, K_k]=\epsilon_{ikl}K_l, & [K_i, K_k]=-\epsilon_{ikl}L_l, \\ [L_i, P_k]=\epsilon_{ikl}P_l, & [K_i, P_k]=\delta_{ik}P_0, & [K_i, P_0]=P_i, \\ [L_i, P_0]=0, & [P_\mu, P_\nu]=0, & [D, P_\mu]=-P_\mu, \\ [L_i, C_k]=\epsilon_{ikl}C_l, & [K_i, C_k]=\delta_{ik}C_0, & [K_i, C_0]=C_i, \\ [L_i, C_0]=0, & [C_\mu, C_\nu]=0, & [D, C_\mu]=C_\mu, \\ [P_0, C_0]=2D, & [P_i, C_k]=-2\delta_{ik}D-2\epsilon_{ikl}L_l, & [D, L_i]=0, \\ [P_0, C_i]=2K_i & [P_i, C_0]=-2K_i, & [D, K_i]=0. \end{array} \qquad (1)$$

Now let us consider a lightlike vector l, write it as a difference between two otherwise arbitrary vectors x and y, and find the maximal subgroup of C(3, 1) leaving the vector space spanned by l invariant. Thus we have

$$l=x-y, \quad l^2=l_0^2-\mathbf{l}^2=(x-y)^2=0, \qquad (2)$$

and we put $l=(\omega, 0, 0, -\omega)$. Let us first consider subgroups of the similitude group Sim(3,1) (Poincaré extended by dilations). Clearly all four translations P_μ leave l invariant, as does its little group E(2), generated by L_3, L_2+K_1, and L_1-K_2. The boost $\exp(aK_3)$ will transform l into $e^a l$, the dilation e^{aD} will transform it into $e^{-a}l$. Thus (K_3-D) will generate transformations leaving l invariant. Now let us consider proper conformal transformations; $\exp(cC)$ will transform the component l_μ as in:

$$l_\mu \to l'_\mu = x'_\mu - y'_\mu = \frac{x_\mu + c_\mu x^2}{\sigma(x)} - \frac{y_\mu + c_\mu y^2}{\sigma(y)}, \qquad (3)$$

where $\sigma(x)$ is the usual conformal factor

$$\sigma(x)=1+2cx+c^2x^2. \qquad (4)$$

The requirement $x'_i - y'_i=0$ for $i=1, 2$ implies that $c_0=-c_3$, $c_1=c_2=0$ in (3). We thus obtain the transformation generated by C_0-C_3 that leaves l invariant. Thus, requiring that the vector space l remain invariant, i.e.,

$$l'=e^\rho l, \qquad (5)$$

we obtain the group Opt(3, 1) and its algebra opt(3,1) (we shall use capital letters for Lie groups and noncapital ones for the corresponding Lie algebras) spanned by the following C(3,1) generators:

$$\{L_3, K_3, L_2+K_1, L_1-K_2, P_0, P_1, P_2, P_3, D, C_0-C_3\}. \qquad (6)$$

If we require that the vector l itself be invariant, i.e., $\rho=0$ in (5), then we must omit $D+K_3$ from (6) and this leads us from Opt(3, 1) to the extended Schrödinger group group Sch_2, leaving the nonrelativistic Schrödinger equation

$$\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi = i\frac{\partial\psi}{\partial t} \qquad (7)$$

invariant.

In this article, as in a previous one,[5] we shall use a different basis for opt(3,1), making its nonrelativistic aspects more evident, namely

$$\begin{array}{llll} p_1=-P_1, & p_2=-P_2, & k_1=-L_2-K_1, & k_2=L_1-K_2, \\ m=\frac{1}{2}(P_0-P_3), & t=\frac{1}{2}(P_0+P_3), & c=\frac{1}{2}(C_0-C_3), & d=D-K_3, \\ j=L_3, & s=-(D+K_3). & & \end{array} \qquad (8)$$

In these notations p_i generate translations, k_i Galilei boosts ($i=1, 2$), m corresponds to the mass, t, c, and d together generate an SL(2,R) group, and j generates O(2) rotations. These nine operators provide a basis for sch_2, the extended Schrödinger algebra. Among them p_i, k_i, m, j, and the time translation t generate

TABLE I. Action of Opt (3, 1) on Minkowski space in light cone coordinates: $x_\pm = x_0 \pm x_3$; a_i, v_i, κ, τ, β, γ, ϕ, and λ are real parameters; $\sigma(x)=1+\beta x$, $x^2=x_+x_- - x_1^2 - x_2^2$.

Transformation	x_+	x_-	x_1	x_2
p_i	x_+	x_-	$x_1+a_1\delta_{i1}$	$x_2+a_2\delta_{i2}$
k_i	$x_+-2v_jx_j-v_jx_-$	x_-	$x_1+v_1x_-\delta_{i1}$	$x_2+v_2x_-\delta_{i2}$
m	$x_++\kappa$	x_-	x_1	x_2
t	x_+	$x_-+\tau$	x_1	x_2
c	$\frac{x_++\beta x^2}{\sigma(x)}$	$\frac{x_-}{\sigma(x)}$	$\frac{x_1}{\sigma(x)}$	$\frac{x_2}{\sigma(x)}$
d	x_+	$e^{2\gamma}x_-$	$e^\gamma x_1$	$e^\gamma x_2$
j	x_+	x_-	$\cos\phi x_1+\sin\phi x_2$	$-\sin\phi x_2 + \cos\phi x_2$
s	$e^{-2\lambda}x_+$	x_-	$e^{-\lambda}x_1$	$e^{-\lambda}x_2$

the extended Galilei group. The dilation operator d and the "conformal" transformation (or "expansion") c were first introduced by Hagen[6] in the context of a nonrelativistic field theory. Finally, the additional dilation s is external to the Schrödinger group.

The nonrelativistic meaning of the group Opt(3,1) is further demonstrated by considering its action on Minkowski space, making use of "light cone coordinates": $x_{\pm}=x_0 \pm x_3$, x_1, x_2. The action of the individual transformations is demonstrated in Table I. The group Sch_2 can be seen to act explicitly as the Galilei group, extended by dilations d and expansions c on a Newtonian $2+1$ space–time, namely the hyperplane $\{x_-, x_1, x_2\}$. Here $\{x_1, x_2\}$ is the "transverse plane" of the infinite momentum frame, and x_- plays the role of time.

An alternative manner of characterizing the optical algebra opt(3,1) is to identify $m=(p_0-p_2)/2$ with a "color" operator[8] (here "color" is used in the sense of the frequency of a massless particle). The algebra opt(3,1) can be characterized as the normalizer of m in the conformal Lie algebra c(3,1), i.e., it satisfies

$$[\mathrm{opt}(3,1), m]=\lambda m, \tag{9}$$

where λ is a constant. More specifically, we have $\lambda=0$ for the subalgebra $\mathrm{sch}_2 \subset \mathrm{opt}(3,1)$ and $\lambda \neq 0$ for the dilation s; thus sch_2 is the centralizer of m in c(3,1).

B. The optical group in O(4,2) and SU(2,2) and its relation to the similitude group

In view of the local isomorphism

$$\mathrm{C}(3,1) \approx \mathrm{SO}_0(4,2)/Z_2 \approx \mathrm{SU}(2,2)/Z_4, \tag{10}$$

where the centers are $Z_2=\{1,-1\}$ and $Z_4=\{i,-1,-i,1\}$, respectively, we can also realize Opt(3,1) as a maximal subgroup of $\mathrm{SO}_0(4,2)$ [the identity component of O(4,2)] or SU(2,2).

Indeed, let us first consider the algebra o(4,2) realized by 6×6 real matrices X, satisfying

$$XK+KX^T=0, \tag{11}$$

where X^T is X transposed. We shall use a realization in which

$$K=\begin{bmatrix} 0 & 0 & J_2 \\ 0 & I_2 & 0 \\ J_2 & 0 & 0 \end{bmatrix}, \quad I_2=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

[rather than the usual one in which K is diagonal with eigenvalues $(1,1,1,1,-1,-1)$]. The matrix X can then be written as

$$X=\begin{bmatrix} a & b & c & d & e & 0 \\ f & g & h & j & 0 & -e \\ l & m & 0 & n & -h & -c \\ p & q & -n & 0 & -j & -d \\ k & 0 & -m & -q & -g & -b \\ 0 & -k & -l & -p & -f & -a \end{bmatrix}. \tag{13}$$

The group Opt(3,1) can now be characterized as the maximal subgroup of $\mathrm{SO}_0(4,2)$, leaving a two-dimensional isotropic (completely lightlike) subspace of the pseudo-Euclidean space $E_{4,2}$ invariant. Indeed, such a subspace can in the realization (11) be written as (the superscript T indicates transposition)

$$f^T=(x,y,0,0,0,0), \tag{14}$$

and the requirement

$$Xf=f', \tag{15}$$

where f' is a vector of the type (14), implies

$$l=m=p=q=k=0 \tag{16}$$

in (13). Let us now use capital letters for a basis of o(4,2), namely, e.g., A will be the element X of (13) with $a=1$ and all other entries $b=c=\cdots=0$. The generators of the conformal group and in particular of Opt(3,1) [according to (8)] can be identified as

$$\begin{aligned} &L_1=(H-M)/\sqrt{2}, && L_2=(J-Q)/\sqrt{2}, && L_3=-N, \\ &K_1=(J+Q)/\sqrt{2}, && K_2=(-H-M)/\sqrt{2}, && K_3=-G, \\ &P_1=\sqrt{2}D, && P_2=-\sqrt{2}C, && P_3=-E-B, \\ &P_0=E-B, && C_0=F-K, && D=-A, \\ &C_1=\sqrt{2}P, && C_2=-\sqrt{2}L, && C_3=-F-K. \end{aligned} \tag{17}$$

A different maximal subalgebra of o(4,2) is obtained by requiring that a one-dimensional $E_{4,2}$ lightlike vector space be left invariant

$$Xg=g', \tag{18}$$

where g and g' are of the form (14) with $y=0$. This is the similitude algebra sim(3,1).

In this article we are simply interested in classifying the subgroups of Opt(3,1) into conjugacy classes under the group Opt(3,1) itself. The list of subgroups will, however, constitute a part of a larger list, namely that of the subgroups of C(3,1) itself. We will hence be interested in establishing (and eventually eliminating) the very significant overlap between subalgebras of sim(3,1) and opt(3,1). Obviously a subalgebra of opt(3,1) will also be contained in sim(3,1) if it leaves a lightlike vector invariant. Taking X in the form (13) and using (16), we see that an opt(3,1) matrix has a lightlike eigenvector (corresponding to a real eigenvalue), if the equation

$$\begin{pmatrix} a & b \\ f & g \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\lambda\begin{pmatrix} x \\ y \end{pmatrix} \tag{19}$$

has real eigenvalues λ. This always happens unless

$$(a-g)^2+4bf<0, \tag{20}$$

which requires that

$$bf<0. \tag{21}$$

Hence an algebra that is contained in opt(3,1) and not in sim(3,1) must contain at least one element with a nonreal eigenvalue in the $\{A, B, F, G\}$ subalgebra. The elements of this type can be written as $B-F+\kappa(A+G)$, where κ is an arbitrary real number. In terms of the opt(3,1) generators (8) such an element is

$$c+t+\kappa s. \tag{22}$$

In other words, a subalgebra of opt(3,1) is not contained in sim(3,1) [i.e., not conjugate under the conformal group to one contained in sim(3,1)] if and only if it contains a rotation $c+t$ or a mixture of a rotation and dilation $c+t+\kappa s$ (with κ real) in its $\{c,t,d,s\}$ part.

The su(2,2) algebra can be realized by complex 4×4 matrices Y satisfying

$$YJ+JY^{+}=0, \tag{23}$$

where Y^{+} is the Hermitian conjugate of Y. For our purposes it is convenient to choose J in the form

$$J=\begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix} \tag{24}$$

rather than as a diagonal matrix with eigenvalues (1, 1, −1, −1). The matrix Y can then be written as

$$Y=\begin{bmatrix} +it & \alpha & \gamma & ia \\ \beta & u-it & ib & -\gamma^* \\ \delta & ic & -u-it & -\alpha^* \\ id & -\delta^* & -\beta^* & -s+it \end{bmatrix}, \tag{25}$$

where the Latin entries are real, the Greek ones are complex, and the star denotes complex conjugation.

The algebra opt(3,1) is this time obtained as the maximal subalgebra of su(2,2) leaving a lightlike complex vector invariant. In the realization (23) the vector f satisfying

$$f^T=(z,0,0,0) \tag{26}$$

is lightlike since $f^{+}Jf=0$ and the condition

$$Yf=f',$$

where f' is also of the form (26) implies

$$\beta=\delta=id=0. \tag{27}$$

The sim(3,1) algebra in this realization leaves a two-dimensional isotropic vector space

$$(u,v,0,0) \tag{28}$$

invariant and hence satisfies $\delta=c=d=0$.

The generators of opt(3,1) can in this realization be chosen to be

$$k_1=\begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&-1\\0&0&0&0\end{bmatrix},\quad k_2=\begin{bmatrix} 0&i&0&0\\0&0&0&0\\0&0&0&i\\0&0&0&0\end{bmatrix},$$

$$p_1=\begin{bmatrix} 0&0&i&0\\0&0&0&i\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad p_2=\begin{bmatrix} 0&0&-1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$t=\begin{bmatrix} 0&0&0&0\\0&0&i&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad c=\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&i&0&0\\0&0&0&0\end{bmatrix},$$

$$d=\begin{bmatrix} 0&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\; j=\frac{i}{2}\begin{bmatrix} 1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix},$$

$$m=\begin{bmatrix} 0&0&0&i\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad s=\begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix}. \tag{29}$$

Let us comment here that all maximal subalgebras of the conformal Lie algebra c(3,1) have been listed in an article devoted to a classification of Abelian sub-algebras of semisimple Lie algebras.[27]

C. General comments on the optical group

The commutation relations for the basis elements (8) of opt(3,1) are given in Table II. Notice that $w_2=\{p_1,p_2,k_1,k_2,m\}$ is a nilpotent ideal and is isomorphic to a Weyl–Heisenberg algebra, where p_1 and p_2 are identified with momenta, k_1 and k_2 with the canonically conjugate coordinates, and m is a constant (the Planck constant). The factor algebra opt$(3,1)/w_2$ $\sim\{t,c,d,j,s\}$ has the form $\mathrm{sl}(2,r)\oplus \mathrm{o}(2)\oplus \mathrm{o}(1,1)$.

The Schrödinger group Sch_2 generated by $\{p_1,p_2,k_1,k_2,m,t,c,d,j\}$ has three Casimir operators, namely,[5]

$$\begin{aligned} &C^{(1)}=m,\quad C^{(2)}=jm-k_1p_2+k_2p_1\\ &C^{(4)}=2(2mt-p_1^2-p_2^2)(2mc-k_1^2-k_2^2)+2(2mc-k_1^2-k_2^2)\\ &\qquad\times(2mt-p_1^2-p_2^2)-(2md+k_1p_1+p_1k_1\\ &\qquad+k_2p_2+p_2k_2)^2 \end{aligned} \tag{30}$$

(for notations see also the caption of Table III).

None of these commute with the additional dilation s and indeed opt(3,1) has no Casimir operators. It does, however, have two rational invariants,[28] lying in the quotient field of the enveloping algebra, namely

$$R_1=(jm-k_1p_2+k_2p_1)/m=C^{(2)}/m,\quad R_2=C^{(4)}/m^2. \tag{31}$$

These invariants can be used to characterize irreducible representations of Opt(3,1) in the same manner as Casimir operators (polynomial invariants), for instance, the zero-mass discrete spin representations of the Poincaré group extended to the conformal group, when restricted to the optical group, are characterized by the eigenvalue zero of R_2 and the eigenvalues of R_1 which correspond exactly to helicity.

3. SUBALGEBRA CLASSIFICATION METHOD

We could classify the subalgebras of opt(3,1) into Opt(3,1) conjugacy classes directly, making use of the fact that opt(3,1) has a five-dimensional invariant subalgebra $w_2\sim\{p_1,p_2,k_1,k_2,m\}$ with the factor algebra $F\equiv \mathrm{opt}(3,1)/w_2\sim\{s,j,c,d,t\}$. Instead we shall profit from the fact that the subalgebras of the (extended) Schrödinger algebra $\mathrm{sch}_2\sim\{j,c,d,t,p_1,p_2,k_1,k_2,m\}$ have already been classified[5] into conjugacy classes under the (extended) Schrödinger group Sch_2 and extend sch_2 and the group Sch_2 by the additional dilation s. The classification both in this article and the previous one[5] is performed with respect to the identity component of the corresponding group Opt(3,1) or Sch_2 (no reflections are included).

TABLE II. Commutation relations for opt(3, 1).

	k_1	k_2	p_1	p_2	m	j	t	c	d	s
k_1	0	0	m	0	0	$-k_2$	p_1	0	$-k_1$	$-k_1$
k_2	0	0	0	m	0	k_1	p_2	0	$-k_2$	$-k_2$
p_1	$-m$	0	0	0	0	$-p_2$	0	$-k_1$	p_1	$-p_1$
p_2	0	$-m$	0	0	0	p_1	0	$-k_2$	p_2	$-p_2$
m	0	0	0	0	0	0	0	0	0	$-2m$
j	k_2	$-k_1$	p_2	$-p_1$	0	0	0	0	0	0
t	$-p_1$	$-p_2$	0	0	0	0	0	d	$2t$	0
c	0	0	k_1	k_2	0	0	$-d$	0	$-2c$	0
d	k_1	k_2	$-p_1$	$-p_2$	0	0	$-2t$	$2c$	0	0
s	k_1	k_2	p_1	p_2	$2m$	0	0	0	0	0

The subalgebras of sch_2 were classified using a straightforward algorithm, consisting of several steps. The algorithm has already been described,[14] as has its application[5] to sch_2, and we shall not go into it here. Let us just state that a list of representatives of Sch_2 conjugacy classes of subalgebras of sch_2 will contain two types of subalgebras—"splitting" and "nonsplitting" ones. For the splitting subalgebras, it is always possible to choose a basis consisting entirely of elements B_i contained in the factor algebra F and elements X_a, contained in the ideal w_2. A nonsplitting subalgebra, on the other hand, is not conjugate under Sch_2 to a splitting one and any basis for a nonsplitting subalgebra will contain at least one element of the type $B+X$ with $B\in F$ and $X\in w_2$ $(B\neq 0,\ X\neq 0)$.

The list of representatives of all Sch_2 conjugacy classes of subalgebras of sch_2 is given in Table 2 of Ref. 5. For the purposes of the present article we shall denote the splitting subalgebras in that list $s_{j,k}$, the nonsplitting ones $\bar{s}_{j,k}$. Here j denotes the dimension of the subalgebra and k enumerates the subalgebras of a given dimension. To obtain a complete list of representatives of Opt(3,1) classes of subalgebras of opt(3,1) we proceed as follows.

1. Take all splitting subalgebras $s_{j,k}$ of the above list. Without any modification they will go over into the opt(3,1) list.

2. Take all nonsplitting subalgebras $\bar{s}_{j,k}$ from the sch_2 list. Those that do not depend on a continuous parameter, e.g., $\{t+k_1, p_1;m\}$ or $\{t+\epsilon m, p_1, p_2;\epsilon=\pm 1\}$ will go over directly into the opt(3,1) list. If a nonsplitting sch_2 subalgebra depends on one or more continuous parameters, contained in the "nonsplitting generators," and connecting the F and w_2 parts of these generators, then the dilation exps must be used to scale a nonzero parameter of this type to some specific chosen value (we usually chose it to be equal to ± 1). An example of this type is the algebra $\{j-\epsilon(c+t)+\alpha(k_2-\epsilon p_1),\ k_1+\epsilon p_2;m\}$ with $\epsilon=\pm 1$ and $\alpha\neq 0$. The operator exps dilates k_i, p_i, and m but commutes with j, c, t and d. Hence in opt(3,1) we can scale α to $\alpha=\pm 1$. If more than one parameter is involved, e.g., $\{j+\alpha m,\ c+t+\beta m\}$ then only one parameter can be scaled. Thus, the subalgebra $\{j+\alpha m,\ c+t+\beta m\}$ in the sch_2 list contributes the subalgebra $\{j+\epsilon m,\ c+t+\beta m\}$, $\{j,\ c+t+\epsilon m\}$, and $\{j,\ c+t\}$ (with $\epsilon=\pm 1$, α and β arbitrary real numbers) in the opt(3,1) list. All algebras obtained from $\bar{s}_{j,k}$ by such a scaling must be included in the opt(3,1) list.

3. Construct all subalgebras of opt(3,1) containing the generator s (in some basis). We obtain representatives of all of these by taking all splitting subalgebras from the sch_2 list and adding the generator s to them. Thus, e.g., the sch_2 subalgebra $\{d;k_1,k_2,p_1,p_2,m\}$ provides the opt(3,1) subalgebra $\{s,d;k_1,k_2,p_1,p_2,m\}$.

4. Construct all subalgebras of opt(3,1) containing a nontrivial "coupling" between the dilation s and an element of sch_2, i.e., a generator of the type

$$s+\alpha B+\beta X,\quad B\in F,\quad X\in w_2,\quad \alpha,\beta\in \mathbb{R},\quad \alpha^2+\beta^2\neq 0. \tag{32}$$

To obtain representatives of all of these algebras, we once more run through the list of all subalgebras $s_{j,k}$ and $\bar{s}_{j,k}$ of sch_2. For each one of them we construct its normlaizer in opt(3,1):

$$\mathrm{nor}\, s_{jk}=\{X\,|\,X\in \mathrm{opt}(3,1),\ [X,s_{j,k}]\subseteq s_{j,k}\}. \tag{33}$$

A new subalgebra to be included in the opt(3,1) list will be obtained in one of two cases:

(a) The normalizer $\mathrm{nor}s_{j,k}$ contains s and some other nonzero element or elements x_i of sch_2, not contained in $s_{j,k}$. The new algebra is then obtained by taking s_{jk}, adding $s+\alpha_i x_i$ to it (α_i are real constants) and then using $\mathrm{Nor}s_{j,k}=\exp(\mathrm{nor}s_{j,k})$ to simplify the element $s+\alpha_i x_i$ as much as possible [$\mathrm{Nor}s_{j,k}$ is a subgroup of Opt(3,1)].

As an example consider the sch_2 subalgebra $\{p_1, p_2, m\}$. Its normalizer in opt(3,1) is $\{s, d, j, t, k_1, k_2, p_1, p_2, m\}$. Thus we obtain the set of subalgebras

$$\{s=s+\alpha_1 d+\alpha_2 j+\alpha_3 t+\alpha_4 k_1+\alpha_5 k_2,\ p_1, p_2, m\}. \quad (34)$$

We now use the normalizer of $\{p_1, p_2, m\}$ in the *group* Opt(3,1) to simplify the element s. The transformation $\exp(-\alpha_3 t/2\alpha_1)$ will eliminate $\alpha_3 t$ if $\alpha_1\neq 0$. If $\alpha_1=0$, then $\exp ad$ will scale α_3 to any chosen number of the same sign as α_3 and hence we need only consider $\alpha_3=0$ or $\alpha_3\equiv\epsilon=\pm 1$. The transformation $\exp(xk_1+yk_2)$ with an appropriate choice of x and y will eliminate k_1, and k_2, unless we have $\alpha_2=0$, $\alpha_1=-1$. In this last case $\exp\phi j$ with an appropriate choice of ϕ will rotate $\alpha_4 k_1+\alpha_5 k_2$ into βk_1 with $\beta=\sqrt{(\alpha_4^2+\alpha_5^2)}\geq 0$ and $\exp\gamma s$ will scale β into 1 (for $\beta>0$.) We thus obtain the subalgebras:

$$\begin{aligned}&\{s+\alpha j+\beta d,\ p_1, p_2, m\}, \quad \alpha^2+\beta^2\neq 0,\\ &\{s+\alpha j+\epsilon t,\ p_1, p_2, m\} \quad \alpha\in\mathbb{R},\ \epsilon=\pm 1, \qquad (35)\\ &\{s-d+k_1,\ p_1, p_2, m\},\end{aligned}$$

and these are the only subalgebras of opt(3,1) provided by the sch_2 subalgebra p_1, p_2, m and having a nontrivial "coupling" with s.

(b) The normalizer of $s_{j,k}$ may not contain s itself but some combination of the type $s+f$, $f\in\mathrm{sch}_2$, $f\notin s_{j,k}$. We take the element, add to it an arbitrary linear combination of all the elements of $\mathrm{nor}s_{j,k}$, not contained in $s_{j,k}$, and then proceed to simplify by $\mathrm{Nor}s_{j,k}$ as in case (a).

As an example consider the algebra $\{t+k_1, p_2, m\}$. Its normalizer is $\{s-\frac{1}{3}d,\ t+k_1, k_2, p_1, p_2, m\}$. Thus we obtain the algebras

$$\{s-\tfrac{1}{3}d+\alpha_1 k_2+\alpha_2 p_1,\ t+k_1, p_2, m\}.$$

The transformation $\exp(\frac{3}{2}\alpha_1 k_2+\frac{3}{4}\alpha_2 p_1)$ will eliminate the $\alpha_1 k_2+\alpha_2 p_1$ terms, and we obtain the opt(3,1) subalgebra

$$\{s-d/3;\ t+k_1, p_2, m\}.$$

The above algorithm provides an exhaustive and nonoverlapping list of representatives of all Opt(3,1) classes of subalgebras of opt(3,1). Hence, any subalgebra of opt(3,1) is conjugate to one in the list, and no two members in the list are mutually conjugate.

4. SUBALGEBRAS OF THE OPTICAL ALGEBRA

A systematic application of the algorithm presented in Sec. 3 provides us with the complete list of representatives of Opt(3,1) classes of subalgebras of opt(3,1), summarized in Tables III–VIII (see end of text).

In Table III we list all subalgebras of dimension $10\geq d\geq 6$. Column 1 provides a name for each algebra. The first subscript denotes the dimension, the second one enumerates different algebras of the same dimension. The ordering is such that we first list separable algebras (they only occur for $d=6$), then nilpotent nonseparable ones and finally nonseparable and nonnilpotent ones. These are so ordered that the dimension of the derived algebra increases as we proceed along the list: perfect Lie algebras, e.g., $r_{8,8}$ are thus at the end of the list for any given dimension. Superscripts, e.g., $r_{8,5}^{\alpha}$ or $r_{7,13}^{\alpha\epsilon}$ indicate that the entry actually represents a set of algebras, different (and mutually nonconjugate) for each value of the parameters. Throughout we have $\epsilon\equiv\pm 1$; the ranges of all other parameters are given in column 3, where we also give a basis for each algebra. If the range of a parameter is not specified, then it is allowed to run through all real numbers ($\alpha\in\mathrm{R}$). The generators to the right of a semicolon in column 3 span the derived algebra. The normalizer $\mathrm{nor}r_{j,k}$ in opt(3,1) of each algebra is given in column 4 and the invariants are listed in column 5. The invariants were calculated by solving a certain set of partial differential equations,[28] and we include nonpolynomial invariants as well as polynomial ones (Casimir operators). Isomorphisms between different algebras in the list are indicated in column 2. Thus, e.g., $r_{9,3}^{\alpha}$ and $r_{9,3}^{-\alpha}$ ($\alpha\neq 0$) are isomorphic to each other [but not conjugate under Opt(3,1); if they were we would have stipulated $\alpha\geq 0$]. Similarly, $r_{6,14}^{-1/\alpha}$ and $r_{6,15}^{\alpha,0}$ are isomorphic to each other for any fixed $\alpha\neq 0$.

In Tables IV–VI, we present all subalgebras with $d=5$, 4, and 3 respectively. In column 1 we introduce a name for each subalgebra. In column 2 we list its isomorphism class, using notations introduced earlier[28] (they follow a classification of low dimensional Lie algebras, essentially due to Mubarakzyanov[29]). No such classification is available for dimension $d\geq 6$. Columns 3, 4, and 5 have the same meaning as in Table II and the same comments apply. The two dimensional subalgebras are listed in Table VII. Column 1 introduces a name, column 2 gives the isomorphism class (Abelian $2A_1$ and non-Abelian A_2). Column 3 lists the generators and range of parameters (if any), and column 4 gives the normalizers. Algebras A_2 have no invariants; both generators are invariants of the $2A_1$ algebras.

The one-dimensional subalgebras are listed in Table VIII. This has only three columns—the name, the generators, and the normalizers.

This completes the classification of all subalgebras of opt(3,1). Subgroups of Opt(3,1) will in general be obtained from these subalgebras by exponentiation. The algebra opt(3,1) contains two commuting rotation operators: j and $c+t$. The exponentiation of a generator of the type $j+\alpha(c+t)$ with α irrational does not lead to a closed connected subgroup (its closure is a two-dimensional subgroup, rather than a one-dimensional one). Thus, to obtain representatives of all closed connected subgroups of Opt(3,1), we proceed as follows. If a considered algebra contains the element $j+\alpha(c+t)$ [or $j+\alpha(c+t)+\epsilon m$, or $j+\alpha(c+t)+\beta s$], we allow the parameter α to be rational only and then exponentiate the algebra. All other subalgebras we exponentiate directly.

5. CONCLUSIONS

The main result of this paper is a complete classification of all subalgebras of the algebra opt(3,1) and of all closed connected subgroups of the group Opt(3,1). These results are summarized in Tables III–VIII.

Several applications of this subgroup classification are in preparation. The first is the completion of a series of articles on the "Subgroups of the Fundamental Groups of Physics,"[14,18] namely a classification of the subgroups of the conformal group of space–time C(3,1). Only subalgebras of opt(3,1), not contained in sim(3,1) will be relevant for an over-all list of subalgebras of c(3,1). These can be directly picked out from Tables III–VIII. Indeed, these "new" subalgebras are characterized by the fact that they contain an element of the type $c+t+\alpha s+\beta j+\gamma_i k_i+\delta_i p_i+\xi m$, where α, β, γ_i, δ_i, and ξ are abritrary real numbers.

A second application is a study of symmetry breaking for the two dimensional time dependent Schrödinger equation. Indeed, we are presently investigating the equation

$$-\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-i\frac{\partial\psi}{\partial t}=F, \tag{36}$$

where F is an arbitrary function of say (x, y, t, ψ, ψ^*) or more generally of $(x, y, t, \psi, \psi^*, \psi_x, \psi_x^*, \psi_y, \psi_y^*, \psi_t, \psi_t^*)$. For each subgroup of the extended Schrödinger group Sch(2) we are looking for the most general "interaction" F, such that (36) should be invariant under the chosen subgroup. A similar problem has already been solved in the one dimensional case.[20] This type of study is relevant, e.g., for parton models involving interacting partons.

The third application is the construction of tensors (electromagnetic potentials and field, metric tensors, etc.) that are invariant under different subgroups of the conformal group.

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TABLE III. Subalgegras of opt(3,1) with dimension $d=10,\ 9,\ 8,\ 7,\ 6$. The notations used for the invariants: $\mathcal{J}=2(jm-k_1p_2+k_2p_1)$, $\mathcal{T}=2mt-p_1^2-p_2^2$, $\mathcal{C}=2mc-k_1^2-k_2^2$, $\mathcal{D}=2md+k_1p_1+p_1k_1+k_2p_2+p_2k_2$, $\mathcal{R}^2=2\mathcal{T}\mathcal{C}+2\mathcal{C}\mathcal{T}-\mathcal{D}^2$.

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{10,1}$	opt(3,1)	$s,j;c,d,t,k_1,k_2,p_1,p_2,m$	self	J/m, R^2/m^2
$r_{9,1}$		$s,j,d;t,k_1,k_2,p_1,p_2,m$	self	J/m
$r_{9,2}$	sch_2	$j;c,d,t,k_1,k_2,p_1,p_2,m$	$r_{10,1}$	m,J,R^2
$r_{9,3}^{\alpha}$	$(\alpha)\sim(-\alpha)$	$s+\alpha j;c,d,t,k_1,k_2,p_1,p_2,m$	$r_{10,1}$	R^2/m^2
$r_{8,1}$		$s,j,d;k_1,k_2,p_1,p_2,m$	self	$J/m,D/m$
$r_{8,2}$		$s,j,t+c;k_1,k_2,p_1,p_2,m$	self	$J/m,(T+C)/m$
$r_{8,3}$		$s,j,t;k_1,k_2,p_1,p_2,m$	$r_{9,1}$	$J/m,T/m$
$r_{8,4}$		$j,d;t,k_1,k_2,p_1,p_2,m$	$r_{9,1}$	m,J
$r_{8,5}^{\alpha}$		$s+\alpha d,j;t,k_1,k_2,p_1,p_2,m$	$r_{9,1}$	J/m, $m^{\alpha-1}T$
$r_{8,6}^{\alpha}$	$(\alpha)\sim(-\alpha)$	$s+\alpha j,d;t,k_1,k_2,p_1,p_2,m$	$r_{9,1}$	none
$r_{8,7}^{\alpha\beta}$	$(\alpha,\beta)\sim(-\alpha,-\beta)$	$s+\alpha j,j+\beta d;t,k_1,k_2,p_1,p_2,m$ $(\beta\neq 0)$	$r_{9,1}$	none
$r_{8,8}$		$;c,d,t,k_1,k_2,p_1,p_2,m$	$r_{10,1}$	m,R^2
$r_{7,1}$		$s,j,d;t,p_1,p_2,m$	self	$mt/(p_1^2+p_2^2)$
$r_{7,2}$		$s-d,k_1,k_2;t,p_1,p_2,m$	$r_{9,1}$	T/m^2
$r_{7,3}$		$s,d;t,k_1,p_1,p_2,m$	self	$(2mt-p_1^2)/p_2^2$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{7,4}$		$j,d;k_1,k_2,p_1,p_2,m$	$r_{8,1}$	$m,J,\mathcal{D}$
$r_{7,5}$		$j,c+t;k_1,k_2,p_1,p_2,m$	$r_{8,2}$	$m,J,C+T$
$r_{7,6}$		$j,t;k_1,k_2,p_1,p_2,m$	$r_{9,1}$	m,J,T
$r_{7,7}^{\alpha\beta}$	$(\alpha,\beta)\sim(-\alpha,-\beta)$	$s+\beta j,j+\alpha(c+t);k_1,k_2,p_1,p_2,m$ $(\alpha\neq 0)$	$r_{8,2}$	$[J+\alpha(C+T)]/m$
$r_{7,8}$		$s,j;k_1,k_2,p_1,p_2,m$	$r_{10,1}$	J/m
$r_{7,9}^{\varepsilon}$	$(\varepsilon=1)\sim(\varepsilon=-1)$	$s+\varepsilon t,j;k_1,k_2,p_1,p_2,m$	$r_{8,3}$	J/m
$r_{7,10}^{\alpha}$		$s+\alpha d,j;k_1,k_2,p_1,p_2,m$ $(\alpha>0)$	$r_{8,1}$	J/m
$r_{7,11}^{\alpha}$	$(\alpha)\sim(-\alpha)$	$s+\alpha j,c+t;k_1,k_2,p_1,p_2,m$	$r_{8,2}$	$(C+T)/m$
$r_{7,12}^{\alpha}$	$(\alpha)\sim(-\alpha)$	$s+\alpha j,t;k_1,k_2,p_1,p_2,m$	$r_{9,1}$	T/m
$r_{7,13}^{\alpha\varepsilon}$	$(\alpha,\varepsilon)\sim(-\alpha,\varepsilon)$	$s+\alpha j,j+\varepsilon t;k_1,k_2,p_1,p_2,m$	$r_{8,3}$	$(J+\varepsilon T)/m$
$r_{7,14}^{\alpha\beta}$	$(\alpha,\beta)\sim(-\alpha,\beta)$	$s+\alpha j,d+\beta j;k_1,k_2,p_1,p_2,m$ $(\beta\geq 0)$	$r_{8,1}$	$(\mathcal{D}+\beta T)/m$
$r_{7,15}^{\alpha}$	$(\alpha)\sim(-\alpha)$	$d+\alpha j;t,k_1,k_2,p_1,p_2,m$	$r_{9,1}$	m
$r_{7,16}^{\alpha\beta}$	$(\alpha,\beta)\sim(-\alpha,\beta)$	$s+\alpha j+\beta d;t,k_1,k_2,p_1,p_2,m$ $(\beta\neq 0;\beta\neq -1$ if $\alpha=0)$	$r_{9,1}$	$m^{\beta-1}T$
$r_{7,17}$		$s;c,d,t,k_1,p_1,m$	self	$-[(2(2mt-p_1^2)(2mc-k_1^2)+2(2mc-k_1^2)(2mt-p_1^2)$ $-(2md+p_1k_1+k_1p_1)]/m$
$r_{6,1}$	$A_1+A_2+A_{3,8}$	$(j)\oplus(s;m)\oplus(;c,d,t)$	self	$j,(c+t)^2-(c-t)^2-d^2$
$r_{6,2}$	$A_1+A_{5,30}^{0}$	$(p_2)\oplus(s+d;t,k_1,p_1,m)$	$r_{7,3}$	$p_2,2mt-p^2$
$r_{6,3}$	$A_1+A_{5,35}^{0,2}$	$(t)\oplus(s,j;p_1,p_2,m)$	$r_{7,1}$	$t,(p_1^2+p_2^2)/m$
$r_{6,4}$	$A_1+A_{5,35}^{0,2}$	$(m)\oplus(d,j;t,p_1,p_2)$	$r_{7,1}$	$m,(p_1^2+p_2^2)/t$
$r_{6,5}^{\varepsilon}$	$A_1+A_{5,37}$	$\{j-\varepsilon(c+t)\}\oplus\{s,j+\varepsilon(c+t);$ $k_1+\varepsilon p_2,k_2-\varepsilon p_1,m\}$	self	$j-\varepsilon(c+t),(C+T+\varepsilon J)/m$
$r_{6,6}$	$A_2+A_{4,12}$	$\{s+d;t\}\oplus\{j,d;p_1,p_2\}$	self	none
$r_{6,7}$	$A_2+A_{4,12}$	$\{s+d;m\}\oplus\{j,d;p_1,p_2\}$	self	none
$r_{6,8}$	$A_{3,4}+A_{3,6}$	$\{s+d;t,m\}\oplus\{j;p_1,p_2\}$	$r_{7,1}$	$tm,\ p_1^2+p_2^2$
$r_{6,9}$	$A_{6,14}^{1}$	$t,k_1,k_2;p_1,p_2,m$	$r_{9,1}$	$m,\ T$
$r_{6,10}^{\varepsilon}$		$s-d,p_1,p_2;k_1,k_2,m$	$r_{8,1}$	none
$r_{6,11}^{\varepsilon}$	$(\varepsilon=1)\sim(\varepsilon=-1)$	$j+\varepsilon(c+t),k_1-\varepsilon p_2,k_2+\varepsilon p_1;$ $k_1+\varepsilon p_2,k_2-\varepsilon p_1,m$	$r_{8,2}$	$m,J+\varepsilon(C+T)$
$r_{6,12}^{\alpha}$		$s+\alpha d,j;t,p_1,p_2,m$ $(\alpha\neq 0,1)$	$r_{7,1}$	$tm^{\alpha},\ (p_1^2+p_2^2)\ m^{\alpha-1}$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{6,13}$		$s-d,k_1;t,p_1,p_2,m$	$r^0_{8,6}$	$p_2/m,(2mt-p_1^2)/m^2$
$r_{6,14}$		$s-d+k_1,k_2;t,p_1,p_2,m$	$r_{7,2}$	T/m^2, $m\exp(-2p_2/m)$
$r^{\alpha\beta}_{6,15}$	$r^{-1/\alpha}_{6,14}\sim r^{\alpha,0}_{6,15}$	$s+\beta j,d+\alpha j;t,p_1,p_2,m$	$r_{7,1}$	$(p_1^2+p_2^2)/mt$, $\left(\frac{p_1+ip_2}{p_1-ip_2}\right)^{i}(p_1^2+p_2^2)^{\alpha}/m^{\alpha+\beta}$
$r_{6,16}$		$s,t;k_1,p_1,p_2,m$	$r_{7,2}$	p_2^2/m, $(2mt-p_1^2)/m$
$r_{6,17}$		$s,d;k_1,p_1,p_2,m$	self	none
$r^{\varepsilon}_{6,18}$	$(\varepsilon=1)\sim(\varepsilon=-1)$	$s,j+\varepsilon(c+t);k_1+\varepsilon p_2,k_2,p_1,m$	self	$(k_2+\varepsilon p_1)/m,(C+T+\varepsilon J)/m$
$r_{6,19}$		$s,d;t,k_1,p_1,m$	self	none
$r_{6,20}$		$d;t,k_1,p_1,p_2,m$	$r_{7,3}$	$m,(2mt-p_1^2)/p_2^2$
$r^{\alpha}_{6,21}$		$s+\alpha d;t,k_1,p_1,p_2,m$ $(\alpha\neq 0,\varepsilon)$	$r_{7,3}$	$p_2^2m^{\alpha-1}$, $(2mt-p_1^2)m^{\alpha-1}$
$r_{6,22}$		$j;k_1,k_2,p_1,p_2,m$	$r_{10,1}$	m,J
$r_{6,23}$		$d;k_1,k_2,p_1,p_2,m$	$r_{8,1}$	m,D
$r_{6,24}$		$c+t;k_1,k_2,p_1,p_2,m$	$r_{8,2}$	$m,C+T$
$r^{\alpha}_{6,25}$		$j+\alpha(c+t);k_1,k_2,p_1,p_2,m$ $(\alpha\neq 0,\varepsilon)$	$r_{8,2}$	$m,J+\alpha(C+T)$
$r^{\varepsilon}_{6,26}$	$(\varepsilon=1)\sim(\varepsilon=-1)$	$j+\varepsilon t;k_1,k_2,p_1,p_2,m$	$r_{8,3}$	$m,J+\varepsilon T$
$r^{\alpha}_{6,27}$		$j+\alpha d;k_1,k_2,p_1,p_2,m$ $(a>0)$	$r_{8,1}$	$m,J+\alpha D$
$r_{6,28}$		$s+\alpha j;k_1,k_2,p_1,p_2,m$	$r_{10,1}$	none
$r^{\alpha,\beta}_{6,29}$		$s+\alpha j+\beta d;k_1,k_2,p_1,p_2,m$ $(\beta>0;\beta\neq 1$ if $\alpha=0)$	$r_{8,1}$	none
$r^{\alpha,\beta}_{6,30}$		$s+\alpha j+\beta(c+t);k_1,k_2,p_1,p_2,m$ $(\beta\neq 0)$	$r_{8,2}$	none
$r^{\alpha,\varepsilon}_{6,31}$		$s+\alpha j+\varepsilon t;k_1,k_2,p_1,p_2,m$	$r^0_{8,4}$	none
$r_{6,32}$		$s-\frac{1}{3}d;t+k_1,k_2,p_1,p_2,m$	self	none
$r_{6,33}$		$;c,d,t,k_1,p_1,m$	$r_{7,17}$	$m,2(2mt-p_1^2)(2mc-k_1^2)+2(2mc-k_1^2)(2mt-p_1^2)-(2md+p_1k_1+k_1p_1)^2$

TABLE IV. Five-dimensional subalgebras.

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{5,1}$	$2A_1+A_{3,4}$	$(p_1)\oplus(p_2)\oplus(s+d;t,m)$	$r_{7,1}$	p_1,p_2,tm
$r_{5,2}$	$2A_1+A_{3,6}$	$(m)\oplus(t)\oplus(j;p_1,p_2)$	$r_{7,1}$	$m,t,p_1^2+p_2^2$
$r_{5,3}$	$2A_1+A_{3,8}$	$(s)\oplus(j)\oplus(;c,d,t)$	self	$s,j,(c+t)^2-(c-t)^2-d^2$
$r_{5,4}$	$2A_1+A_{3,8}$	$(j)\oplus(m)\oplus(;c,d,t)$	$r_{6,1}$	$j,m(c+t)^2-(c-t)^2-d^2$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{5,5}$	A_1+2A_2	$(j)\oplus(s;m)\oplus(d;t)$	self	j
$r_{5,6}$	$A_2+A_{3,3}$	$(s+d;m)\oplus(d;p_1,p_2)$	$r_{6,7}$	p_1/p_2
$r_{5,7}$	$A_2+A_{3,6}$	$(s+d;m)\oplus(j;p_1,p_2)$	$r_{6,7}$	$p_1^2+p_2^2$
$r_{5,8}$	$A_2+A_{3,6}$	$(s+d;t)\oplus(j;p_1,p_2)$	$r_{6,6}$	$p_1^2+p_2^2$
$r_{5,9}^{\alpha}$	$A_2+A_{3,7}^{\lvert\alpha\rvert}$	$(s+d;m)\oplus(j+\alpha d;p_1,p_2)\quad(\alpha\neq 0)$	$r_{6,7}$	$(p_1^2+p_2^2)\left(\frac{p_1+ip_2}{p_1-ip_2}\right)^{i\alpha}$
$r_{5,10}^{\alpha}$	$A_2+A_{3,7}^{\lvert\alpha\rvert}$	$(s+d;t)\oplus(j+\alpha s;p_1,p_2)\quad(\alpha\neq 0)$	$r_{6,6}$	$(p_1^2+p_2^2)\left(\frac{p_1-ip_2}{p_1+ip_2}\right)^{i\alpha}$
$r_{5,11}^{\alpha}$	$A_2+A_{3,8}$	$(s+\alpha j;m)\oplus(;c,d,t)$	$r_{6,1}$	$(c+t)^2-(c-t)^2-d^2$
$r_{5,12}$	$A_1+A_{4,1}$	$(p_2)\oplus(t,k_1;p_1,m)$	$r_{7,3}$	p_2,m,p_1^2-2mt
$r_{5,13}$	$A_1+A_{4,5}^{\frac{1}{2},\frac{1}{2}}$	$(m)\oplus(d;t,p_1,p_2)$	$r_{7,1}$	$m,(p_1^2+p_2^2)/t,p_1/p_2$
$r_{5,14}$	$A_1+A_{4,5}^{\frac{1}{2},\frac{1}{2}}$	$(t)\oplus(s;p_1,p_2,m)$	$r_{7,1}$	$t,p_1^2/m,p_2^2/m$
$r_{5,15}^{\alpha}$	$A_1+A_{4,6}^{2\lvert\alpha\rvert,\lvert\alpha\rvert}$	$(t)\oplus(j+\alpha s;p_1,p_2,m)\quad(\alpha\neq 0)$	$r_{7,1}$	$t,(p_1^2+p_2^2)\left(\frac{p_1-ip_2}{p_1+ip_2}\right)^{i\alpha},\ m/(p_1^2+p_2^2)$
$r_{5,16}^{\alpha}$	$A_1+A_{4,6}^{2\lvert\alpha\rvert,\lvert\alpha\rvert}$	$(m)\oplus(j+\alpha d;t,p_1,p_2)\quad(\alpha\neq 0)$	$r_{7,1}$	$m,t/(p_1^2+p_2^2),\ (p_1^2+p_2^2)\left(\frac{p_1+ip_2}{p_1-ip_2}\right)^{i\alpha}$
$r_{5,17}$	$A_1+A_{4,9}^{0}$	$(p_2)\oplus(s+d,p_1;k_1,m)$	$r_{6,17}$	p_2
$r_{5,18}^{\varepsilon}$	$A_1+A_{4,9}^{1}$	$(j-\varepsilon(c+t))\oplus(s;k_1+\varepsilon p_2,k_2-\varepsilon p_1,m)$	$r_{6,5}^{\varepsilon}$	$j-\varepsilon(c+t)$
$r_{5,19}^{\varepsilon}$	$A_1+A_{4,10}$	$(j-\varepsilon(c+t))\oplus$ $(j+\varepsilon(c+t);k_1+\varepsilon p_2,k_2-\varepsilon p_1,m)$	$r_{6,5}^{\varepsilon}$	$j-\varepsilon(c+t),m,C+T+\varepsilon J$
$r_{5,20}^{\varepsilon}$	$A_1+A_{4,10}$	$(k_2+\varepsilon p_1)\oplus$ $(j+\varepsilon(c+t);k_1+\varepsilon p_2,k_2-\varepsilon p_1,m)$	$r_{7,7}^{\varepsilon,0}$	$k_2+\varepsilon p_1,m,C+T+\varepsilon J$
$r_{5,21}^{\varepsilon,\alpha}$	$A_1+A_{4,11}^{\lvert\alpha\rvert}$	$(j-\varepsilon(c+t))\oplus$ $(j+\alpha s;k_1+\varepsilon p_2,k_2-\varepsilon p_1,m)\quad(\alpha\neq 0)$	$r_{6,5}^{\varepsilon}$	$j-\varepsilon(c+t)$
$r_{5,22}$	$A_1+A_{4,12}$	$(s+d)\oplus(s,j;p_1,p_2)$	self	$s+d$
$r_{5,23}$	$A_1+A_{4,12}$	$(m)\oplus(d,j;p_1,p_2)$	$r_{6,7}$	m
$r_{5,24}$	$A_1+A_{4,12}$	$(t)\oplus(s,j;p_1,p_2)$	$r_{6,6}$	t
$r_{5,25}$	$A_{5,4}$	$k_1,k_2,p_1,p_2;m$	$r_{10,1}$	m
$r_{5,26}$	$A_{5,5}$	$t+k_1,k_2,p_1;p_2,m$	$r_{7,16}^{0,-1/3}$	m
$r_{5,27}$	$A_{5,7}^{1,1,1}$	$s-d;t,p_1,p_2,m$	$r_{9,1}$	$t/m,p_1/m,p_2/m$
$r_{5,28}^{\alpha}$	$A_{5,7}^{x,x,z}$	$s+\alpha d;t,p_1,p_2,m\quad(\alpha\neq 0,\varepsilon)$ $\left(\begin{array}{l} x=\frac{1-\alpha}{2},\ z=-\alpha \text{ if } \lvert\alpha\rvert<1 \\ x=\frac{\alpha-1}{2\alpha},\ z=-\frac{1}{\alpha} \text{ if } \lvert\alpha\rvert>1 \end{array}\right)$	$r_{7,1}$	$m^{\alpha}t,\ p_1^2m^{\alpha-1},\ p_1/p_2$
$r_{5,29}$	$A_{5,11}^{1}$	$s-d+k_1;t,p_1,p_2,m$	$r_{7,2}$	$T/m^2,\ m\exp(-2p_1/m),p_2/m$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\alpha,\beta}_{5,30}$	$A^{x,y,z}_{5,13}$	$j+\alpha s+\beta d;t,p_1,p_2,m$ $(\alpha\beta\neq 0)$ $\left[x=-\frac{\alpha}{\beta},\ y=\frac{\beta-\alpha}{2\beta},\ z=\frac{1}{2\lvert\beta\rvert}\ \text{if}\lvert\alpha\rvert\le\lvert\alpha\rvert\right]$ $\left[x=-\frac{\beta}{\alpha},\ y=\frac{\alpha-\beta}{2\alpha},\ z=\frac{1}{2\lvert\alpha\rvert}\ \text{if}\lvert\alpha\rvert>\lvert\beta\rvert\right]$	$r_{7,1}$	$t^{\alpha}m^{\beta}, (p_1^2+p_2^2)^{\alpha}m^{\beta-\alpha}$, $m(\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\varepsilon}_{5,31}$	$A^{2,1}_{5,19}$	$s;k_1+\varepsilon p_2,k_2,p_1,m$	$r^{\varepsilon}_{6,18}$	$(k_2+\varepsilon p_1)^2/m$
$r_{5,32}$	$A^{2,1}_{5,19}$	$s;k_1,p_1,p_2,m$	$r_{7,3}$	p_2^2/m
$r_{5,33}$	$A^{3/2,1}_{5,19}$	$s-\frac{1}{3}d;t+k_1,p_1,p_2,m$	self	p_2^3/m^2
$r_{5,34}$	$A^{1,1}_{5,19}$	$s-d;k_1;p_1,p_2,m$	$r^{0,0}_{7,14}$	p_2/m
$r_{5,35}$	$A^{0,1}_{5,19}$	$d;k_1,p_1,p_2,m$	$r_{6,17}$	m
$r^{\alpha}_{5,36}$	$A^{x,y}_{5,19}$	$s+\alpha d;k_1,p_1,p_2,m$ $(\alpha\neq 0,\varepsilon)$ $\left[x=\frac{2}{1+\alpha},\ y=\frac{1-\alpha}{1+\alpha}\ \text{if}\ 0<\alpha\right]$ $\left[x=\frac{2}{1-\alpha},\ y=1\ \text{if}\ \alpha<0\right]$	$r_{6,17}$	$p_2^2m^{\alpha-1}$
$r_{5,37}$	$A^{1}_{5,20}$	$s-d+k_2,k_1;p_1,p_2,m$	$r_{6,9}$	$m\exp(-\frac{2p_2}{m})$
$r^{\varepsilon}_{5,38}$	$A^{1}_{5,23}$	$s+\varepsilon t;k_1,p_1,p_2,m$	$r_{6,16}$	p_2^2/m
$r^{\alpha,\varepsilon}_{5,39}$	$A^{\lvert\varepsilon\rvert/2,\lvert\alpha\rvert/2}_{5,25}$	$j+\varepsilon(c+t)+\alpha s;k_1+\varepsilon p_2,k_2,p_1,m$ $(\alpha\neq 0)$	$r_{6,18}$	$(k_2+\varepsilon p_1)^2/m$
$r^{\varepsilon}_{5,40}$	$A^{0,1}_{5,26}$	$j+\varepsilon(c+t)+k_1-\varepsilon p_2,k_2+\varepsilon p_1;$ $k_1+\varepsilon p_2,k_2-\varepsilon p_1,m$	$r^{\varepsilon}_{6,10}$	m
$r_{5,41}$	$A^{2}_{5,30}$	$s-\frac{1}{3}d;t+k_1,k_2,p_2,m$	self	$[p_2^2-2m(t+k_1)]^3/m^4$
$r^{\varepsilon}_{5,42}$	$A^{1}_{5,30}$	$s,t;k_1+\varepsilon p_2,p_1,m$	self	$(p_1^2-2mt)/m$
$r_{5,43}$	$A^{0}_{5,30}$	$s+d;t,k_1,p_1,m$	$r_{7,3}$	p_1^2-2mt
$r_{5,44}$	$A^{0}_{5,30}$	$s+d+p_2;t,k_1,p_1,m$	$r_{6,2}$	p_1^2-2mt
$r_{5,45}$	$a^{-1}_{5,30}$	$d;t,k_1,p_1,m$	$r_{6,19}$	m
$r^{\alpha}_{5,46}$	$A^{x}_{5,30}$	$s+\alpha d;t,k_1,p_1,m$ $(\alpha\neq\varepsilon\ ;\ x=(1-\alpha)/(1+\alpha))$	$r_{6,19}$	$(p_1^2-2tm)m^{\alpha-1}$
$r_{5,47}$	$A^{0}_{5,32}$	$s-d;t,k_1,p_1,m$	$r_{6,19}$	$(p_1^2-2mt)/m^2$
$r_{5,48}$	$A^{\frac{1}{2},\frac{1}{2}}_{5,33}$	$s,d;t,p_1,m$	self	tm/p_1^2
$r_{5,49}$	$A^{0,1}_{5,33}$	$s,d;t,p_1,p_2$	$r_{6,6}$	p_1/p_2
$r_{5,50}$	$A^{-1,1}_{5,33}$	$s,d;k_1,p_2,m$	self	k_1p_2/m
$r_{5,51}$	$A^{0,2}_{5,35}$	$s,j;p_1,p_2,m$	$r_{7,1}$	$(p_1^2+p_2^2)/m$
$r^{\alpha,\varepsilon}_{5,52}$	$A^{0,2}_{5,35}$	$s+\alpha t,j+\varepsilon t;p_1,p_2,m$	$r_{6,3}$	$(p_1^2+p_2^2)/m$
$r^{\varepsilon}_{5,33}$	$A^{0,2}_{5,35}$	$s+\varepsilon t,j;p_1,p_2,m$	$r_{6,3}$	$(p_1^2+p_2^2)/m$
$r_{5,54}$	$A^{0,2}_{5,35}$	$d,j;p_1,p_2,t$	$r_{7,1}$	$(p_1^2+p_2^2)/t$
$r^{\varepsilon,\alpha}_{5,55}$	$A^{0,2}_{5,35}$	$d+\varepsilon m,j+\alpha m;p_1,p_2,t$	$r_{6,4}$	$(p_1^2+p_2^2)/t$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\varepsilon}_{5,56}$	$A^{0,2}_{5,35}$	$d, j+\varepsilon m; p_1, p_2, t$	$r_{6,4}$	$(p_1^2+p_2^2)/t$
$r^{\varepsilon}_{5,57}$	$A^{0,1}_{5,35}$	$s-d, j; t+\varepsilon m, p_1, p_2$	self	$(p_1^2+p_2^2)/(t+\varepsilon m)^2$
$r^{\varepsilon}_{5,58}$	$A^{0,2/(1-\alpha)}_{5,35}$	$s+\alpha d, j; p_1, p_2, m \quad (\alpha\neq 0,1)$	$r_{6,7}$	$(p_1^2+p_2^2)m^{\alpha-1}$
$r^{\alpha}_{5,59}$	$A^{0,2\alpha/(1-\alpha)}_{5,35}$	$s+\alpha d, j; p_1, p_2, t \quad (\alpha\neq 0,1)$	$r_{6,6}$	$(p_1^2+p_2^2)^{\alpha} t^{1-\alpha}$
$r^{\alpha}_{5,60}$	$A^{2\lvert\alpha\rvert,0}_{5,35}$	$j+\alpha s, d; p_1, p_2, m \quad (\alpha\neq 0)$	$r_{6,7}$	$m(\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\alpha}_{5,61}$	$A^{2\lvert\alpha\rvert,2}_{5,35}$	$j+\alpha s, d; p_1, p_2, t \quad (\alpha\neq 0)$	$r_{6,6}$	$[t/(p_1^2+p_2^2)](\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\alpha,\beta}_{5,62}$	$A^{x,y}_{5,35}$	$s+\alpha d, j+\beta(s+d); p_1, p_2, m$ $(\alpha\neq 1, \beta\neq 0;\ x=2\lvert\beta\rvert,\ y=\frac{2}{1-\alpha})$	$r_{6,7}$	$m^{\alpha-1}(p_1^2+p_2^2)(\frac{p_1+ip_2}{p_1-ip_2})^{i(1-\alpha)\beta}$
$r^{\alpha,\beta}_{5,63}$	$A^{x,y}_{5,35}$	$s+\alpha d, j+\beta(s+d); p_1, p_2, t$ $(\alpha\neq 1, \beta\neq 0;\ x=2\lvert\beta\rvert, y=\frac{2\alpha}{\alpha-1})$	$r_{6,6}$	$t^{(\alpha-1)}(p_1^2+p_2^2)^{-\alpha}(\frac{p_1+ip_2}{p_1-ip_2})^{i(\alpha-1)\beta}$
$r^{\alpha}_{5,64}$	$A_{5,36}$	$s, d; k_1, p_1+\alpha p_2, m$	self	$(k_1(p_1+\alpha p_2)+(p_1+\alpha p_2)k_1+2md)/m$
$r^{\alpha,\varepsilon}_{5,65}$	$A_{5,37}$	$s+\alpha(c+t), j; k_1+\varepsilon p_2, k_2-\varepsilon p_1, m$	$r^{\varepsilon}_{6,5}$	$((k_1+\varepsilon p_2)^2+(k_2-\varepsilon p_1)^2-4\varepsilon mj)/m$
$r^{\alpha}_{5,66}$	$A_{5,37}$	$s, c+t; k_1+\alpha p_2, p_1-\alpha k_2, m (0\leq\lvert\alpha\rvert<1)$	self	$((k_1+\alpha p_2)^2+(p_1-\alpha k_2)^2-2(1+\alpha^2)m(c+t))/m$
$r^{\alpha,\varepsilon}_{5,67}$	$A_{5,37}$	$s+\alpha j, c+t; k_1+\varepsilon p_2, p_1-\varepsilon k_2, m$	$r^{\varepsilon}_{6,5}$	$((k_1+\varepsilon p_2)^2+(p_1-\varepsilon k_2)^2-4m(c+t))/m$
$r^{\alpha,\beta}_{5,68}$	$A_{5,37}$	$s+\alpha j, j+\beta(c+t); k_1+\varepsilon p_2, k_2-\varepsilon p_1, m$ $(\beta\neq 0, -\varepsilon)$	$r^{\varepsilon}_{6,5}$	$((k_1+\varepsilon p_2)^2+(k_2-\varepsilon p_1)^2-\frac{4\varepsilon m}{1+\varepsilon\beta}(j+\beta(c+t)))/m$

TABLE V. Four-dimensional subalgebras.

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{4,1}$	$4A_1$	t, m, p_1, p_2	$r_{9,1}$	t, m, p_1, p_2
$r_{4,2}$	$2A_1+A_2$	$(p_1)\oplus(p_2)\oplus(s+d;t)$	$r_{6,6}$	p_1, p_2
$r_{4,3}$	$2A_1+A_2$	$(p_1)\oplus(p_2)\oplus(s+d;m)$	$r_{6,7}$	p_1, p_2
$r_{4,4}$	$2A_1+A_2$	$(j)\oplus(m)\oplus(d;t)$	$r_{5,5}$	j, m
$r_{4,5}$	$2A_1+A_2$	$(j)\oplus(t)\oplus(s;m)$	$r_{5,5}$	j, t
$r_{4,6}$	$2A_1+A_2$	$(j)\oplus(s)\oplus(d;t)$	self	j, s
$r_{4,7}$	$2A_1+A_2$	$(j)\oplus(t+c)\oplus(s;m)$	self	$j, t+c$
$r_{4,8}$	$2A_1+A_2$	$(j)\oplus(d)\oplus(s;m)$	self	j, d
$r^{\alpha}_{4,9}$	$2A_2$	$(d;t)\oplus(s+\alpha j;m)$	$r_{5,5}$	none

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\alpha}_{4,10}$	$2A_2$	$(j+\alpha d;t)\oplus(s+\beta j;m)$ $(\alpha\neq 0)$	$r_{5,5}$	none
$r_{4,11}$	$2A_2$	$(d+s;t)\oplus(s;p_1)$	self	none
$r_{4,12}$	$2A_2$	$(d+s;k_2)\oplus(d-s;p_1)$	self	none
$r_{4,13}$	$2A_2$	$(d+s;m)\oplus(d;p_1)$	self	none
$r_{4,14}$	$A_1+A_{3,1}$	$(p_2)\oplus(k_1,p_1;m)$	$r^{0}_{8,6}$	p_2,m
$r_{4,15}$	$A_1+A_{3,1}$	$(p_2)\oplus(t+k_1,p_1;m)$	$r^{0,-1/3}_{7,16}$	p_2,m
$r_{4,16}$	$A_1+A_{3,1}$	$(k_2+\varepsilon p_1)\oplus(k_1+\varepsilon p_2,k_2-\varepsilon p_1;m)$	$r^{\varepsilon,0}_{7,7}$	$k_2+\varepsilon p_1,m$
$r^{\varepsilon}_{4,17}$	$A_1+A_{3,1}$	$(j-\varepsilon(c+t))\oplus(k_1+\varepsilon p_2,k_2-\varepsilon p_1;m)$	$r^{\varepsilon}_{6,5}$	$j-\varepsilon(c\ t),m$
$r_{4,18}$	$A_1+A_{3,2}$	$(p_2)\oplus(s+d+p_1;k_1,m)$	$r_{5,17}$	$p_2,m\ \exp(2k_1/m)$
$r_{4,19}$	$A_1+A_{3,3}$	$(t)\oplus(s;p_1,p_2)$	$r_{6,6}$	$t,p_1/p_2$
$r_{4,20}$	$A_1+A_{3,3}$	$(m)\oplus(d;p_1,p_2)$	$r_{6,7}$	$m,p_1/p_2$
$r_{4,21}$	$A_1+A_{3,3}$	$(p_2)\oplus(s+d;k_1,m)$	$r_{6,17}$	$p_2,k_1/m$
$r_{4,22}$	$A_1+A_{3,3}$	$(j)\oplus(s-d;t,m)$	$r_{5,5}$	$j,t/m$
$r_{4,23}$	$A_1+A_{3,3}$	$(s+d)\oplus(d;p_1,p_2)$	$r_{5,22}$	$s+d,p_1/p_2$
$r_{4,24}$	$A_1+A_{3,4}$	$(p_1)\oplus(s+d;t,m)$	$r^{0}_{6,14}$	p_1,tm
$r_{4,25}$	$A_1+A_{3,4}$	$(p_1)\oplus(s+d+p_2;t,m)$	$r_{5,1}$	p_1,tm
$r_{4,26}$	$A_1+A_{3,4}$	$(j)\oplus(s+d;t,m)$	$r_{5,5}$	j,tm
$r_{4,27}$	$A_1+A_{3,4}$	$(m)\oplus(d;k_1,p_2)$	$r_{5,50}$	m,k_1p_2
$r_{4,28}$	$A_1+A^{1/2}_{3,5}$	$(m)\oplus(d;t,p_1)$	$r_{5,48}$	$m,p_1^2/t$
$r_{4,29}$	$A_1+A^{1/2}_{3,5}$	$(t)\oplus(s;p_1,m)$	$r_{5,48}$	$t,p_1^2/m$
$r^{\varepsilon}_{4,20}$	$A_1+A^{1/2}_{3,5}$	$(j-\varepsilon(c+t))\oplus(s;k_1+\varepsilon p_2,m)$	self	$j-\varepsilon(c+t)$, $(k_1+\varepsilon p_1)^2/m$
$r^{\alpha}_{4,31}$	$A_1+A^{x}_{3,5}$	$(j)\oplus(s+\alpha d;t,m)$ ($x=\alpha$ if $0<\|\alpha\|<1$, $x=1/\alpha$ if $\|\alpha\|>1$)	$r_{5,5}$	j,tm^{α}
$r_{4,32}$	$A_1+A_{3,6}$	$(t)\oplus(j;p_1,p_2)$	$r_{7,1}$	$t,p_1^2+p_2^2$
$r_{4,33}$	$A_1+A_{3,6}$	$(m)\oplus(j;p_1,p_2)$	$r_{7,1}$	$m,p_1^2+p_2^2$
$r^{\varepsilon}_{4,34}$	$A_1+A_{3,6}$	$(m)\oplus(j+\varepsilon t;p_1,p_2)$	$r_{6,3}$	$m,p_1^2+p_2^2$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\epsilon}_{4,35}$	$A_1+A_{3,6}$	$(t)\oplus(j+\epsilon m;p_1,p_2$	$r_{6,4}$	$t,p_1^2+p_2^2$
$r^{\epsilon}_{4,36}$	$A_1+A_{3,6}$	$(t+\epsilon m)\ \ (j;p_1,p_2)$	$r^{-1}_{6,12}$	$t+\epsilon m,p_1^2+p_2^2$
$r^{\epsilon}_{4,37}$	$A_1+A_{3,6}$	$(t+\epsilon m)\oplus(j+\tilde{\epsilon}m;p_1,p_2)\ \ (\tilde{\epsilon}=\pm 1)$	$r_{5,2}$	$t+\epsilon m,p_1^2\ p_2^2$
$r_{4,38}$	$A_1+A_{3,6}$	$(s+d)\oplus(j;p_1,p_2)$	$r_{5,22}$	$s+d,p_1^2+p_2^2$
$r^{\alpha}_{4,39}$	$A_1+A^{\|\alpha\|}_{3,7}$	$(t)\oplus(j+\alpha s;p_1,p_2)\qquad(\alpha\neq 0)$	$r_{6,6}$	$t,(p_1^2\ p_2^2)(\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\alpha}_{4,40}$	$A_1+A^{\|\alpha\|}_{3,7}$	$(m)\oplus(j+\alpha d;p_1,p_2)\qquad(\alpha\neq 0)$	$r_{6,7}$	$m,(p_1^2+p_2^2)(\frac{p_1+ip_2}{p_1-ip_2})^{i\alpha}$
$r^{\alpha}_{4,41}$	$A_1+A^{\|\alpha\|}_{3,7}$	$(s+d)\oplus(j+\alpha d;p_1,p_2)\qquad(\alpha\neq 0)$	$r_{5,22}$	$s+d,(p_1^2+p_2^2)(\frac{p_1+ip_2}{p_1-ip_2})^{i\alpha}$
$r_{4,42}$	$A_1+A_{3,8}$	$(j)\oplus(;c,d,t)$	$r_{6,1}$	$j,(c+t)^2-(c-t)^2-d^2$
$r_{4,43}$	$A_1+A_{3,8}$	$(m)\oplus(;c,d,t)$	$r_{6,1}$	$m,(c+t)^2-(c-t)^2-d^2$
$r^{\alpha}_{4,44}$	$A_1+A_{3,8}$	$(s+\alpha j)\oplus(;c,d,t)$	$r_{5,3}$	$s+\alpha j,(c+t)^2-(c-t)^2-d^2$
$r^{\epsilon}_{4,35}$	$A_1+A_{3,8}$	$(j+\epsilon m)\oplus(;c,d,t)$	$r_{5,4}$	$j+\epsilon m,(c+t)^2-c-t)^2-d^2$
$r_{4,46}$	$A_{4,1}$	$k_1,t;p_1,m$	$r_{7,3}$	m,p_1^2-2mt
$r^{\epsilon}_{4,47}$	$A_{4,1}$	$k_1+\epsilon p_2,t;p_1,m$	$r_{6,16}$	m,p_1^2-2mt
$r_{4,48}$	$A_{4,1}$	$t+k_1,k_2;p_2,m$	$r_{6,32}$	$m,p_2^2-2m(t+k_1)$
$r_{4,49}$	$A_{4,1}$	$t+k_1,k_2+\epsilon p_1;p_2,m$	$r_{5,26}$	$m,(p_2-\epsilon m)^2-2m(t+k_1)$
$r_{4,50}$	$A^1_{4,2}$	$s-d+k_1;p_1,p_2,m$	$r^{0,-1}_{6,29}$	$m\ \exp(-2p_1/m)$
$r_{4,51}$	$A_{4,4}$	$s-d+k_1;t,p_1,m$	$r_{5,47}$	$m\ \exp(-2p_1/m),(p_1^2-2mt)/m^2$
$r_{4,52}$	$A^{1/3,2/3}_{4,5}$	$s-\frac{1}{3}d;t+k_1,p_2,m$	self	$(t+k_1)^3/m,p_2^3/m^2$
$r_{4,53}$	$A^{1/2,1/2}_{4,5}$	$d;t,p_1,p_2$	$r_{7,1}$	$p_1^2/t,p_2^2/t$
$r_{4,54}$	$A^{1/2,1/2}_{4,5}$	$s;p_1,p_2,m$	$r_{7,1}$	$p_1^2/m,p_2^2/m$
$r^{\epsilon}_{4,55}$	$A^{1/2,1/2}_{4,5}$	$s+\epsilon t;p_1,p_2,m$	$r_{6,3}$	$p_1^2/m,p_2^2/m$
$r^{\epsilon}_{4,56}$	$A^{1/2,1/2}_{4,5}$	$d+\epsilon m;t,p_1,p_2$	$r_{6,4}$	$p_1^2/t,p_2^2/t$
$r_{4,57}$	$A^{1,1}_{4,5}$	$s-d;p_1,p_2,m$	$r_{8,1}$	$p_1/m,\ p_2/m$
$r_{4,58}$	$A^{1,1}_{4,5}$	$s-d;t,p_1,m$	$r_{6,19}$	p_1/m
$r^{\epsilon}_{4,59}$	$A^{1,1}_{4,5}$	$s-d;t+\epsilon m,p_1,p_2$	$r_{5,57}$	$(t+\epsilon m)/p_1,p_2/p_1$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\alpha}_{4,60}$	$A^{x,y}_{4,5}$	$s+\alpha d;t,p_1,p_2\quad(\alpha\neq 0,1)$ $(x,y)=(\frac{\alpha-1}{2\alpha},\frac{\alpha-1}{2\alpha})$ if $\alpha\leq -1$ or $\frac{1}{3}\leq\alpha$ $(x,y)=(\frac{2\alpha}{\alpha-1},1)$ if $-1<\alpha<\frac{1}{3}$	$r_{6,6}$	$p_1^{2\alpha}t^{1-\alpha},\ p_2^{2\alpha}t^{1-\alpha}$
$r^{\alpha}_{4,61}$	$A^{x,y}_{4,5}$	$s+\alpha d;p_1,p_2,m\quad(\alpha\neq 0,\pm 1)$ $(x,y)=(\frac{1-\alpha}{2},\frac{1-\alpha}{2})$ if $-1\leq\alpha\leq 3$ $(x,y)=(\frac{2}{1-\alpha},1)$ if $\alpha<-1$ or $3<\alpha$	$r_{6,7}$	$p_1^2m^{1-\alpha},p_2^2m^{1-\alpha}$
$r^{\alpha}_{4,62}$	$A^{x,y}_{4,5}$	$s+\alpha d;t,p_1,m\quad(\alpha\neq 0)$ $(x,y)=(-\frac{1}{\alpha},\frac{\alpha-1}{2})$ if $\alpha<-1$ or $1<\alpha$ $(x,y)=(-\alpha,\frac{1-\alpha}{2})$ if $-1<\alpha<1$	$r_{5,48}$	$tm^{\alpha},p_1^2m^{1-\alpha}$
$r^{\alpha}_{4,63}$	$A^{x,y}_{4,5}$	$s+\alpha d;k_1,p_2,m\quad(\alpha\neq 1)$ $(x,y)=(\frac{1-\alpha}{2},\frac{1+\alpha}{2})$ if $0\leq\alpha<1$ $(x,y)=(\frac{1-\alpha}{1+\alpha},\frac{2}{1+\alpha})$ if $1<\alpha$	$r_{5,50}$	$p_2^2m^{\alpha-1},k_1^2/m^{1+\alpha}$
$r^{\alpha}_{4,64}$	$A^{2\lvert\alpha\rvert,\lvert\alpha\rvert}_{4,6}$	$j+\alpha d;t,p_1,p_2\quad(\alpha\neq 0)$	$r_{7,1}$	$t/(p_1^2+p_2^2),t(\frac{p_1+ip_2}{p_1-ip_2})^{i\alpha}$
$r^{\alpha}_{4,65}$	$A^{2\lvert\alpha\rvert,\lvert\alpha\rvert}_{4,6}$	$j+\alpha s;p_1,p_2,m\quad(\alpha\neq 0)$	$r_{7,1}$	$m/(p_1^2\ p_2^2),\ m(\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\alpha,\varepsilon}_{4,66}$	$A^{2\lvert\alpha\rvert,\lvert\alpha\rvert}_{4,6}$	$j+\alpha s+\varepsilon t;p_1,p_2,m\quad(\alpha\neq 0)$	$r_{6,3}$	$m/(p_1^2+p_2^2),m(\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\alpha,\varepsilon}_{4,67}$	$A^{2\lvert\alpha\rvert,\lvert\alpha\rvert}_{4,6}$	$j+\alpha d+\varepsilon m;t,p_1,p_2\quad(\alpha\neq 0)$	$r_{6,4}$	$t/(p_1^2\ p_2^2),t(\frac{p_1+ip_2}{p_1-ip_2})^{i\alpha}$
$r^{\alpha,\varepsilon}_{4,68}$	$A^{2\lvert\alpha\rvert,\lvert\alpha\rvert}_{4,6}$	$j+\alpha(s-d);t+\varepsilon m,p_1,p_2\quad(\alpha\neq 0)$	$r^{\varepsilon}_{5,57}$	$(t+\varepsilon m)^2/(p_1^2\ p_2^2),t(\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\alpha,\beta}_{4,69}$	$A^{x,y}_{4,6}$	$j+\alpha s+\beta d;t,p_1,p_2$ $(\alpha\neq 0,\beta\neq 0)$ $(x,y)=(-2\beta,\alpha-\beta)$ if $\alpha\geq\beta$ $(x,y)=(2\beta,\beta-\alpha)$ if $\alpha<\beta$	$r_{6,6}$	$t^{\alpha-\beta}(p_1^2\ p_2^2)^{\beta},t(\frac{p_1\ ip_2}{p_1-ip_2})^{i\beta}$
$r^{\alpha,\beta}_{4,70}$	$A^{x,y}_{4,6}$	$j+\alpha s+\beta d;p_1,p_2,m$ $(\alpha\neq 0,\beta\neq 0)$ $(x,y)=(2\alpha,\alpha-\beta)$ if $\alpha\geq\beta$ $(x,y)=(-2\alpha,\beta-\alpha)$ if $\alpha<\beta$	$r_{6,7}$	$m^{\beta-\alpha}(p_1^2+p_2^2)^{\alpha},m(\frac{p_1-ip_2}{p_1+ip_2})^{i\alpha}$
$r^{\alpha}_{4,71}$	$A_{4,7}$	$s+\alpha t;k_1+\varepsilon p_2,p_1,m\quad(\alpha\neq 0)$	$r^{\varepsilon}_{5,42}$	none
$r^{\varepsilon}_{4,72}$	$A_{4,7}$	$s+\varepsilon t;k_1,p_1,m$	$r^{0}_{5,46}$	none

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\alpha}_{4,73}$	$A_{4,8}$	$d;k_1,p_1+\alpha p_2,m$	$r^{\alpha}_{5,64}$	$m,2md+k_1(p_1\ \alpha p_2)+(p_1+\alpha p_2)k_1$
$r^{\alpha}_{4,74}$	$A^{0}_{4,9}$	$s+d;k_1,p_1+\alpha p_2,m$	$r_{6,17}$	none
$r^{\alpha}_{4,75}$	$A^{0}_{4,9}$	$s+d+p_2;k_1,p_1+\alpha p_2,m$	$r_{5,17}$	none
$r^{\alpha}_{4,76}$	$A^{1/2}_{4,9}$	$s-\frac{1}{3}d;t+k_1,p_1+\alpha p_2,m$	self	none
$r_{4,77}$	$A^{1}_{4,9}$	$s;k_1,p_1,m$	$r_{7,17}$	none
$r^{\alpha,\varepsilon}_{4,78}$	$A^{1}_{4,9}$	$s+\alpha(j-\varepsilon(c+t));k_1+\varepsilon p_2,k_2-\varepsilon p_1,m$	$r^{\varepsilon}_{6,5}$	none
$r^{\varepsilon}_{4,79}$	$A^{1}_{4,9}$	$s;k_1+\varepsilon p_2,p_1,m$	$r^{\varepsilon}_{5,42}$	none
$r^{\alpha}_{4,80}$	$A^{1}_{4,9}$	$s;k_1+\alpha p_2,p_1-\alpha k_2,m\quad (0<\lvert\alpha\rvert<1)$	$r^{\alpha}_{5,66}$	none
$r^{\alpha,\beta}_{4,81}$	$A^{x}_{4,9}$	$s+\alpha d;k_1,p_1+\beta p_2,m$ $(\alpha\neq\varepsilon,\ \alpha\geq 0;\ \text{if}\ \beta=0,\ \alpha\beta\neq 0)$ $\left(x=\frac{1-\alpha}{1\ \alpha}\ \text{if}\ \alpha>C\right.$ $\left.x\ \frac{1\ \alpha}{1-\alpha}\ \text{if}\ \alpha\ \ C\right)$	$r^{\beta}_{5,64}$	none
$r^{\varepsilon}_{4,82}$	$A_{4,10}$	$j+\varepsilon(c+t);k_1+\varepsilon p_2,k_2-\varepsilon p_1,m$	$r_{8,2}$	$m,(k_1+\varepsilon p_2)^2+(k_2-\varepsilon p_1)^2-2m\ (j+\varepsilon(c+t))$
$r^{\alpha,\varepsilon}_{4,83}$	$A_{4,10}$	$j+\alpha(c+t);k_1+\varepsilon p_2,k_2-\varepsilon p_1,m$ $(\lvert\alpha\rvert\neq 1)$	$r^{\varepsilon}_{6,5}$	$m,(k_1+\varepsilon p_2)^2+(k_2-\varepsilon p_1)^2-\frac{4\varepsilon}{\varepsilon+\alpha}m(j+\alpha(c+t))$
$r^{\varepsilon}_{4,84}$	$A_{4,10}$	$c+t;k_1+\varepsilon p_2,p_1-\varepsilon k_2,m$	$r^{\varepsilon}_{6,5}$	$m,(k_1+\varepsilon p_2)^2+(p_1-\varepsilon k_2)^2-4m(c+t)$
$r^{\varepsilon}_{4,85}$	$A_{4,10}$	$j+\varepsilon(c+t)+k_1-\varepsilon p_2;$ $k_1+\varepsilon p_2,k_2-\varepsilon p_1,m$	$r^{\varepsilon}_{6,11}$	$m,(k_1+\varepsilon p_2)^2+(k_2-\varepsilon p_1)^2-2m\varepsilon(j+\varepsilon(c+t)+k_1-\varepsilon p_2)$
$r^{\alpha}_{4,86}$	$A_{4,10}$	$c+t;k_1+\alpha p_2,p_1-\alpha k_2,m\quad (\lvert\alpha\rvert<1)$	$r^{\lvert\alpha\rvert}_{5,66}$	$m,(k_1+\alpha p_2)^2+(p_1-\alpha k_2)^2-2(1+\alpha^2)m(c+t)$
$r^{\alpha,\beta,\varepsilon}_{4,87}$	$A^{x}_{4,11}$	$s+\alpha j+\beta(c+t);k_1+\varepsilon p_2,k_2-\varepsilon p_1,m$ $(\alpha+\varepsilon\beta\neq 0)$ $(x=\frac{1}{\lvert\alpha+\varepsilon\beta\rvert})$	$r^{\varepsilon}_{6,5}$	none
$r^{\alpha}_{4,88}$	$A^{\alpha}_{4,11}$	$c+t+\alpha s;k_1+\alpha p_2,p_1-\alpha k_2,m\ (0<\alpha<1)$	$r^{\alpha}_{5,66}$	none
$r_{4,89}$	$A_{4,12}$	$s,j;p_1,p_2$	$r_{6,6}$	none
$r_{4,90}$	$A_{4,12}$	$d,j;p_1,p_2$	$r_{6,7}$	none
$r^{\alpha}_{4,91}$	$A_{4,12}$	$s+\alpha d,j;p_1,p_2\quad (\alpha\neq 0,1)$	$r_{5,22}$	none
$r^{\alpha,\beta}_{4,92}$	$A_{4,12}$	$j+\alpha s,d+\beta j;p_1,p_2\quad (\alpha\beta\neq 0)$	$r_{5,22}$	none
$r^{\varepsilon}_{4,93}$	$A_{4,12}$	$d,j+\varepsilon m;p_1,p_2$	$r_{5,23}$	none
$r^{\alpha,\varepsilon}_{4,94}$	$A_{4,12}$	$d+\varepsilon m,j+\alpha m;p_1,p_2$	$r_{5,23}$	none

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\varepsilon}_{4,95}$	$A_{4,12}$	$s+\varepsilon t, j; p_1, p_2$	$r_{5,24}$	none
$r^{\alpha,\varepsilon}_{4,96}$	$A_{4,12}$	$s+\alpha t, j+\varepsilon t; p_1, p_2$	$r_{5,24}$	none

TABLE VI. Three-dimensional subalgebras.

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{3,1}$	$3A_1$	p_1, p_2, m	$r_{9,1}$	p_1, p_2, m
$r_{3,2}$	$3A_1$	p_1, p_2, t	$r_{7,1}$	p_1, p_2, t
$r_{3,3}$	$3A_1$	p_1, m, t	$r_{7,3}$	p_1, m, t
$r_{3,4}$	$3A_1$	k_1, p_2, m	$r^{0,0}_{7,14}$	k_1, p_2, m
$r^{\varepsilon}_{3,5}$	$3A_1$	$p_1, p_2, t+\varepsilon m$	$r^{-1}_{6,12}$	$p_1, p_2, t+\varepsilon m$
$r_{3,6}$	$3A_1$	$p_2, m, t+k_1$	$r^{0}_{6,32}$	$p_2, m, t+k_1$
$r_{3,7}$	$3A_1$	j, t, m	$r_{5,5}$	j, t, m
$r^{\varepsilon}_{3,8}$	$3A_1$	$j-\varepsilon(c+t), k_1+\varepsilon p_2, m$	$r^{\varepsilon}_{5,18}$	$j-\varepsilon(c+t), k_1+\varepsilon p_2, m$
$r_{3,9}$	$3A_1$	$s+d, p_1, p_2$	$r_{5,22}$	$s+d, p_1, p_2$
$r_{3,10}$	$3A_1$	s, j, t	$r_{4,6}$	s, j, t
$r_{3,11}$	$3A_1$	$j, c+t, m$	$r_{4,7}$	$j, c+t, m$
$r_{3,12}$	$3A_1$	j, d, m	$r_{4,8}$	j, d, m
$r_{3,13}$	$3A_1$	$s, j, c+t$	self	$s, j, c+t$
$r_{3,14}$	$3A_1$	s, j, d	self	s, j, d
$r_{3,15}$	A_1+A_2	$(j)\oplus(s;m)$	$r_{6,1}$	j
$r_{3,16}$	A_1+A_2	$(j)\oplus(d;t)$	$r_{5,5}$	j
$r^{\alpha}_{3,17}$	A_1+A_2	$(m)\oplus(d+\alpha j;t)$	$r_{5,5}$	m
$r^{\alpha}_{3,18}$	A_1+A_2	$(t)\oplus(s+\alpha j;m)$	$r_{5,5}$	t
$r^{\varepsilon}_{3,19}$	A_1+A_2	$(j)\oplus(s+\varepsilon t;m)$	$r_{4,5}$	j
$r_{3,20}$	A_1+A_2	$(p_1)\oplus(s+d;m)$	$r_{5,6}$	p_1
$r_{3,21}$	A_1+A_2	$(p_1)\oplus(s+d;t)$	$r_{5,49}$	p_1

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{3,22}$	A_1+A_2	$(p_1)\oplus(s+d+p_2;m)$	$r_{4,3}$	p_1
$r^{\varepsilon}_{3,23}$	A_1+A_2	$(j)\oplus(d+\varepsilon m;t)$	$r_{4,4}$	j
$r^{\alpha,\varepsilon}_{3,24}$	A_1+A_2	$(j+\varepsilon m)\oplus(d+\alpha m;t)$	$r_{4,4}$	$j+\varepsilon m$
$r^{\alpha,\varepsilon}_{3,25}$	A_1+A_2	$(j+\varepsilon t)\oplus(s+\alpha j;m)$	$r_{4,5}$	$j+\varepsilon t$
$r^{\alpha}_{3,26}$	A_1+A_2	$(j)\oplus(d+\alpha s;t)$ $(\alpha\neq 0)$	$r_{4,6}$	j
$r^{\alpha,\beta}_{3,27}$	A_1+A_2	$(s+\alpha j)\oplus(d+\beta s;t)$	$r_{4,6}$	$s+\alpha j$
$r^{\alpha,\beta}_{3,28}$	A_1+A_2	$(j+\alpha(c+t))\oplus(s+\beta j;m)$ $(\alpha\neq 0)$	$r_{4,7}$	$j+\alpha(c+t)$
$r^{\alpha}_{3,29}$	A_1+A_2	$(j)\oplus(s+\alpha(c+t);m)$	$r_{4,7}$	j
$r^{\alpha}_{3,30}$	A_1+A_2	$(c+t)\oplus(s+\alpha j;m)$	$r_{4,7}$	$c+t$
$r_{3,31}$	A_1+A_2	$(p_1)\oplus(s+d+p_2;t)$	$r_{4,2}$	p_1
$r^{\alpha,\beta}_{3,32}$	A_1+A_2	$(j+\alpha d)\oplus(s+\beta j;m)$ $(\alpha>0)$	$r_{4,8}$	$j+\alpha d$
$r^{\alpha}_{3,33}$	A_1+A_2	$(d)\oplus(s+\alpha j;m)$	$r_{4,8}$	d
$r_{3,34}$	A_1+A_2	$(t)\oplus(s;p_1)$	$r_{4,11}$	t
$r_{3,35}$	A_1+A_2	$(p_1)\oplus(s+d;k_2)$	$r_{4,12}$	p_2
$r_{3,36}$	A_1+A_2	$(m)\oplus(d;p_1)$	$r_{4,13}$	m
$r^{\varepsilon}_{3,37}$	A_1+A_2	$(j)\oplus(s-d;t+\varepsilon m)$	self	j
$r^{\varepsilon}_{3,38}$	A_1+A_2	$(j+\varepsilon(c+t))\oplus(s;k_1-\varepsilon p_2)$	self	$j+\varepsilon(c+t)$
$r_{3,39}$	A_1+A_2	$(s+d)\oplus(s-d;p_1)$	self	$s+d$
$r^{\varepsilon}_{3,40}$	$A_{3,1}$	$k_1,p_1;m$	$r^{0}_{9,3}$	m
$r^{\varepsilon}_{3,41}$	$A_{3,1}$	$k_1+\varepsilon p_2,k_2-\varepsilon p_1;m$	$r_{8,2}$	m
$r^{\alpha}_{3,42}$	$A_{3,1}$	$k_1+\alpha p_2,p_1-\alpha k_2;m$ $(0<\|\alpha\|<1)$	$r^{0}_{7,11}$	m
$r^{\varepsilon}_{3,43}$	$A_{3,1}$	$p_1+\varepsilon k_2,p_2;m$	$r^{0}_{7,12}$	m
$r^{\alpha}_{3,44}$	$A_{3,1}$	$k_1,p_2+\alpha p_1;m$ $(0<\alpha)$	$r^{0,0}_{7,14}$	m
$r^{\alpha}_{3,45}$	$A_{3,1}$	$k_2+\alpha(t+k_1),p_2;m$ $(\alpha\neq 0)$	$r_{6,32}$	m
$r^{\varepsilon}_{3,46}$	$A_{3,1}$	$j-\varepsilon(c+t)+k_1+\varepsilon p_2,k_2-\varepsilon p_1;m$	$r^{\varepsilon}_{4,17}$	m

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r^{\alpha}_{3,47}$	$A_{3,2}$	$s+d+p_1+\alpha p_2;k_1,m$	$r_{5,17}$	$m\ \exp(-2k_1/m)$
$r_{3,48}$	$A_{3,3}$	$s;p_1,p_2$	$r_{6,6}$	p_1/p_2
$r_{3,49}$	$A_{3,3}$	$d;p_1,p_2$	$r_{6,7}$	p_1/p_2
$r_{3,50}$	$A_{3,3}$	$s+d;k_1,m$	$r_{6,17}$	k_1/m
$r^{\alpha}_{3,51}$	$A_{3,3}$	$s-d+\alpha j;t,m$	$r_{5,5}$	t/m
$r_{3,52}$	$A_{3,3}$	$s+d+p_2;k_1,m$	$r_{5,17}$	k_1/m
$r^{\alpha}_{3,53}$	$A_{3,3}$	$s+\alpha d;p_1,p_2$ $(\alpha\neq 0,1)$	$r_{5,22}$	p_1/p_2
$r^{\varepsilon}_{3,54}$	$A_{3,3}$	$d+\varepsilon m;p_1,p_2$	$r_{5,23}$	p_1/p_2
$r^{\varepsilon}_{3,55}$	$A_{3,3}$	$s+\varepsilon t;p_1,p_2$	$r_{5,24}$	p_1/p_2
$r_{3,56}$	$A_{3,3}$	$s-d;p_1,t$	$r_{4,11}$	p_1/t
$r_{3,57}$	$A_{3,3}$	$s;k_1,p_2$	$r_{4,12}$	k_1/p_2
$r^{\varepsilon}_{3,58}$	$A_{3,3}$	$s-d;t+\varepsilon m,p_1$	self	$(t+\varepsilon m)/p_1$
$r_{3,59}$	$A_{3,4}$	$s+d;t,m$	$r_{7,1}$	tm
$r_{3,60}$	$A_{3,4}$	$s+d+p_1;t,m$	$r_{5,1}$	tm
$r^{\alpha}_{3,61}$	$A_{3,4}$	$s+d+\alpha j;t,m$ $(\alpha\neq 0)$	$r_{5,5}$	tm
$r_{3,62}$	$A_{3,4}$	$d;k_1,p_2$	$r_{5,50}$	k_1p_2
$r_{3,63}$	$A_{3,4}$	$s+\frac{1}{3}d;t,p_1$	$r_{4,11}$	tp_1
$r_{3,64}$	$A_{3,4}$	$s+3d;p_1,m$	$r_{4,13}$	p_1m
$r^{\varepsilon}_{3,65}$	$A_{3,4}$	$d+\varepsilon m;k_1,p_2$	$r_{4,27}$	k_1p_2
$r_{3,66}$	$A^{1/3}_{3,5}$	$s-\frac{1}{3}d;t+k_1,m$	self	$(t+k_1)^3/m$
$r_{3,67}$	$A^{1/2}_{3,5}$	$d;t,p_1$	$r_{5,48}$	p_1^2/t
$r_{3,68}$	$A^{1/2}_{3,5}$	$s;p_1,m$	$r_{5,48}$	p_1^2/m
$r^{\varepsilon}_{3,69}$	$A^{1/2}_{3,5}$	$d+\varepsilon m;t,p_1$	$r_{4,28}$	p_1^2/t
$r^{\varepsilon}_{3,70}$	$A^{1/2}_{3,5}$	$s+\varepsilon t;p_1,m$	$r_{4,29}$	p_1^2/m
$r^{\alpha,\varepsilon}_{3,71}$	$A^{1/2}_{3,5}$	$s+\alpha(j-\varepsilon(c+t));k_1+\varepsilon p_2,m$	$r^{\varepsilon}_{4,30}$	$(k_1+\varepsilon p_2)^2/m$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER	INVARIANTS
$r_{3,72}$	$A_{3,5}^{1/2}$	$s-\frac{1}{3}d;t+k_1,p_2$	self	$p_2(t+k_1)^{-2}$
$r_{3,73}^{\alpha,\beta}$	$A_{3,5}^{x}$	$s+\alpha d+\beta j;t,m$ $\begin{pmatrix} x=-\alpha \text{ if } 0<\|\alpha\|<1; \\ x=-1/\alpha \text{ if } 1<\|\alpha\| \end{pmatrix}$	$r_{5,5}$	tm^{α}
$r_{3,74}^{\alpha}$	$A_{3,5}^{x}$	$s+\alpha d;t,p_1$ $\begin{pmatrix} x=\frac{2\alpha}{\alpha-1} \text{ if } -1<\alpha<\frac{1}{3}, \ \alpha\neq 0; \\ x=\frac{\alpha-1}{2\alpha} \text{ if } \alpha<-1 \text{ or } \frac{1}{3}<\alpha,\alpha\neq 1 \end{pmatrix}$	$r_{4,11}$	$t^{1-\alpha}p_1^{2\alpha}$
$r_{3,75}^{\alpha}$	$A_{3,5}^{x}$	$s+\alpha d;k_1,p_2$ $\begin{pmatrix} x=\frac{1-\alpha}{1+\alpha} \text{ if } 0<\alpha,\alpha\neq 1; \\ x=\frac{1+\alpha}{1-\alpha} \text{ if } \alpha<0,\alpha\neq -1 \end{pmatrix}$	$r_{4,12}$	$k_1^{\alpha-1}\ p_2^{\alpha+1}$
$r_{3,76}^{\alpha}$	$A_{3,5}^{x}$	$s+\alpha d;p_1,m$ $\begin{pmatrix} x=\frac{1-\alpha}{2} \text{ if } -1<\alpha<3,\alpha\neq 0,1; \\ x=\frac{2}{1-\alpha} \text{ if } \alpha<-1 \text{ or } 3<\alpha \end{pmatrix}$	$r_{4,13}$	$m^{\alpha-1}\ p_1^2$
$r_{3,77}$	$A_{3,6}$	$j;p_1,p_2$	$r_{7,1}$	$p_1^2+p_2^2$
$r_{3,78}^{\varepsilon}$	$A_{3,6}$	$j+\varepsilon t;p_1,p_2$	$r_{6,3}$	$p_1^2+p_2^2$
$r_{3,79}^{\varepsilon}$	$A_{3,6}$	$j+\varepsilon m;p_1,p_2$	$r_{6,4}$	$p_1^2+p_2^2$
$r_{3,80}^{\varepsilon,\varkappa}$	$A_{3,6}$	$j+\varepsilon t+\varkappa m;p_1,p_2$ $(\varkappa=\pm 1)$	$r_{5,2}$	$p_1^2+p_2^2$
$r_{3,81}^{\alpha}$	$A_{3,6}$	$j+\alpha(s+d);p_1,p_2$ $(\alpha\neq 0)$	$r_{5,22}$	$p_1^2+p_2^2$
$r_{3,82}^{\alpha}$	$A_{3,7}^{\|\alpha\|}$	$j+\alpha s;p_1,p_2$ $(\alpha\neq 0)$	$r_{6,6}$	$\left(p_1^2+p_2^2\right)\left(\frac{p_1-ip_2}{p_1+ip_2}\right)^{i\alpha}$
$r_{3,83}^{\alpha}$	$A_{3,7}^{\|\alpha\|}$	$j+\alpha d;p_1,p_2$ $(\alpha\neq 0)$	$r_{6,7}$	$\left(p_1^2+p_2^2\right)\left(\frac{p_1+ip_2}{p_1-ip_2}\right)^{i\alpha}$
$r_{3,84}^{\alpha,\beta}$	$A_{3,7}^{x}$	$j+\alpha s+\beta d;p_1,p_2$ $(\alpha\beta\neq 0,\alpha\neq\beta,x\|\alpha-\beta\|)$	$r_{5,22}$	$\left(p_1^2+p_2^2\right)\left(\frac{p_1-ip_2}{p_1+ip_2}\right)^{i(\alpha-\beta)}$
$r_{3,85}^{\alpha,\varepsilon}$	$A_{3,7}^{\|\alpha\|}$	$j+\alpha d+\varepsilon m;p_1,p_2$ $(\alpha\neq 0)$	$r_{5,23}$	$\left(p_1^2+p_2^2\right)\left(\frac{p_1+ip_2}{p_1-ip_2}\right)^{i\alpha}$
$r_{3,86}^{\alpha,\varepsilon}$	$A_{3,7}^{\|\alpha\|}$	$j+\alpha s+\varepsilon t;p_1,p_2$ $(\alpha\neq 0)$	$r_{5,24}$	$\left(p_1^2+p_2^2\right)\left(\frac{p_1-ip_2}{p_1+ip_2}\right)^{i\alpha}$
$r_{3,87}$	$A_{3,8}$	$;c,d,t$	$r_{6,6}$	$(c+t)^2-(c-t)^2-d^2$

TABLE VII. Two-dimensional subalgebras.

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER
$r_{2,1}$	$2A_1$	p_1,m	$r_{8,6}^{0}$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER
$r_{2,2}$	$2A_1$	p_1, p_2	$r_{7,1}$
$r_{2,3}$	$2A_1$	t, m	$r_{7,1}$
$r^{\varepsilon}_{2,4}$	$2A_1$	$k_2+\varepsilon p_1, m$	$r^{\varepsilon,o}_{7,7}$
$r_{2,5}$	$2A_1$	j, m	$r_{6,1}$
$r^{\varepsilon}_{2,6}$	$2A_1$	$j-\varepsilon(c+t), m$	$r^{\varepsilon}_{6,5}$
$r_{2,7}$	$2A_1$	t, p_1	$r^{o}_{6,14}$
$r_{2,8}$	$2A_1$	s, j	$r_{5,3}$
$r_{2,9}$	$2A_1$	j, t	$r_{5,5}$
$r^{\varepsilon}_{2,10}$	$2A_1$	$t+\varepsilon m, p_1$	$r_{5,27}$
$r_{2,11}$	$2A_1$	$t+k_1, m$	$r_{5,33}$
$r_{2,12}$	$2A_1$	k_1, p_2	$r_{5,50}$
$r^{\varepsilon}_{2,13}$	$2A_1$	$j+\varepsilon m, t$	$r_{4,4}$
$r^{\varepsilon}_{2,14}$	$2A_1$	$j+\varepsilon t, m$	$r_{4,5}$
$r^{\varepsilon}_{2,15}$	$2A_1$	$s+\alpha j, t$	$r_{4,6}$
$r_{2,16}$	$2A_1$	$j, c+t$	$r_{4,7}$
$r^{\alpha}_{2,17}$	$2A_1$	$c+t+\alpha j, m \quad (\alpha\neq\varepsilon)$	$r_{4,7}$
$r_{2,18}$	$2A_1$	j, d	$r_{4,8}$
$r^{\alpha}_{2,19}$	$2A_1$	$d+\alpha j, m \quad (\alpha\geq 0)$	$r_{4,8}$
$r^{\varepsilon}_{2,20}$	$2A_1$	$j-\varepsilon(c+t)+k_1+\varepsilon p_2, m$	$r^{\varepsilon}_{4,17}$
$r^{\varepsilon}_{2,21}$	$2A_1$	$j, t+\varepsilon m$	$r_{4,22}$
$r_{2,22}$	$2A_1$	$s+d, p_1$	$r_{4,23}$
$r^{\varepsilon}_{2,23}$	$2A_1$	$j-\varepsilon(c+t),\ k_1+\varepsilon p_2$	$r^{\varepsilon}_{4,30}$
$r_{2,24}$	$2A_1$	$t+k_1, p_2$	$r_{4,52}$
$r^{\varepsilon,\varkappa}_{2,25}$	$2A_1$	$j+\varkappa m, t+\varepsilon m \quad (\varkappa=\pm 1)$	$r_{3,7}$
$r^{\varepsilon,\varkappa}_{2,26}$	$2A_1$	$j-\varepsilon(c+t)+\varkappa m, k_1+\varepsilon p_2 \quad (\varkappa=\pm 1)$	$r_{3,8}$
$r_{2,27}$	$2A_1$	$s+d+p_2, p_1$	$r_{3,9}$
$r^{\varepsilon}_{2,28}$	$2A_1$	$s+\varepsilon t, j$	$r_{3,10}$
$r^{\alpha,\varepsilon}_{2,29}$	$2A_1$	$s+\alpha j, j+\varepsilon t$	$r_{3,10}$
$r^{\varepsilon,\alpha}_{2,30}$	$2A_1$	$j+\varepsilon m, c+t+\alpha m$	$r_{3,11}$
$r^{\varepsilon}_{2,31}$	$2A_1$	$j, c+t+\varepsilon m$	$r_{3,11}$

NAME	ISOMORPHISM CLASS	GENERATORS	NORMALIZER
$r^{\varepsilon,\alpha}_{2,32}$	$2A_1$	$j+\varepsilon m, d+\alpha m$ $(\alpha\geq 0)$	$r_{3,12}$
$r_{2,33}$	$2A_1$	$j, d+m$	$r_{3,12}$
$r^{\alpha}_{2,34}$	$2A_1$	$s+\alpha(c+t), j$ $(\alpha\neq 0)$	$r_{3,13}$
$r^{\alpha,\beta}_{2,35}$	$2A_1$	$s+\alpha j, c+t+\beta j$	$r_{3,13}$
$r^{\alpha}_{2,36}$	$2A_1$	$s+\alpha d, j$ $(\alpha>0)$	$r_{3,14}$
$r^{\alpha}_{2,37}$	$2A_1$	$s+\alpha j, d$	$r_{3,14}$
$r^{\alpha,\beta}_{2,38}$	$2A_1$	$s+\alpha j, j+\beta d$ $(\beta>0)$	$r_{3,14}$
$r^{\alpha}_{2,39}$	A_2	$s+\alpha j; m$	$r_{6,1}$
$r_{2,40}$	A_2	$s+d; t$	$r_{6,6}$
$r_{2,41}$	A_2	$s+d; m$	$r_{6,7}$
$r^{\alpha}_{2,42}$	A_2	$d+\alpha j; t$	$r_{5,5}$
$r_{2,43}$	A_2	$s+d+p_1; t$	$r_{4,2}$
$r_{2,44}$	A_2	$s+d+p_1; m$	$r_{4,3}$
$r^{\varepsilon,\alpha}_{2,45}$	A_2	$d+\alpha j+\varepsilon m; t$	$r_{4,4}$
$r^{\varepsilon,\alpha}_{2,46}$	A_2	$s+\alpha j+\varepsilon t; m$	$r_{4,5}$
$r^{\alpha,\beta}_{2,47}$	A_2	$d+\alpha s+\beta j; t$ $(\alpha\neq 0; \alpha\neq 1 \text{ if } \beta=0)$	$r_{4,6}$
$r^{\alpha,\beta}_{2,48}$	A_2	$s+\alpha j+\beta(c+t); m$ $(\beta\neq 0)$	$r_{4,7}$
$r^{\alpha,\beta}_{2,49}$	A_2	$s+\alpha j+\beta d; m$ $(\beta>0; \beta\neq 1 \text{ if } \alpha=0)$	$r_{4,8}$
$r_{2,50}$	A_2	$s; p_1$	$r_{4,11}$
$r_{2,51}$	A_2	$s-d; p_2$	$r_{4,12}$
$r_{2,52}$	A_2	$d; p_1$	$r_{4,13}$
$r^{\varepsilon}_{2,53}$	A_2	$s+\varepsilon t; p_1$	$r_{3,34}$
$r_{2,54}$	A_2	$s+d+p_1, k_2$	$r_{3,35}$
$r^{\varepsilon}_{2,55}$	A_2	$d+\varepsilon m; p_1$	$r_{3,36}$
$r^{\varepsilon,\alpha}_{2,56}$	A_2	$s-d+\alpha j; t+\varepsilon m$	$r^{\varepsilon}_{3,37}$
$r^{\varepsilon,\alpha}_{2,57}$	A_2	$s+\alpha j(j+\varepsilon(c+t)); k_1-\varepsilon p_2$	$r^{\varepsilon}_{3,38}$
$r^{\varepsilon}_{2,58}$	A_2	$s+\alpha d; p_1$ $(\alpha\neq 0,\varepsilon)$	$r_{3,39}$
$r_{2,59}$	A_2	$s-\frac{1}{3}d; t+k_1$	self

TABLE VIII. One-dimensional subalgebras.

NAME	GENERATORS	NORMALIZER
$r_{1,1}$	m	$r_{10,1}$
$r_{1,2}$	t	$r_{7,1}$
$r_{1,3}$	p_1	$r_{7,3}$
$r_{1,4}$	j	$r_{6,1}$
$r^{\varepsilon}_{1,5}$	$j-\varepsilon(c+t)$	$r^{\varepsilon}_{6,5}$
$r^{\varepsilon}_{1,6}$	$t+\varepsilon m$	$r^{-1}_{6,12}$
$r^{\varepsilon}_{1,7}$	$k_2+\varepsilon p_1$	$r^{\varepsilon}_{6,18}$
$r^{\alpha}_{1,8}$	$s+\alpha j$	$r_{5,3}$
$r^{\varepsilon}_{1,9}$	$j+\varepsilon m$	$r_{5,4}$
$r^{\varepsilon,\varkappa}_{1,10}$	$j-\varepsilon(c+t)+\varkappa m \quad (\varkappa=\pm 1)$	$r^{\varepsilon}_{5,19}$
$r_{1,11}$	$s+d$	$r_{5,22}$
$r^{\varepsilon}_{1,12}$	$j+\varepsilon t$	$r_{4,5}$
$r^{\alpha}_{1,13}$	$c+t+\alpha j \quad (\alpha\neq\varepsilon)$	$r_{4,7}$
$r^{\alpha}_{1,14}$	$d+\alpha j \quad (\alpha\geq 0)$	$r_{4,8}$
$r_{1,15}$	$t+k_1$	$r_{4,52}$
$r^{\varepsilon,\varkappa}_{1,16}$	$j+\varepsilon t+\varkappa m \quad (\varkappa=\pm 1)$	$r_{3,7}$
$r^{\varepsilon}_{1,17}$	$j-\varepsilon(c+t)+k_1+\varepsilon p_2$	$r^{\varepsilon}_{3,8}$
$r_{1,18}$	$s+d+p_1$	$r_{3,9}$
$r^{\varepsilon,\alpha}_{1,19}$	$s+\alpha j+\varepsilon t$	$r_{3,10}$
$r^{\varepsilon,\alpha}_{1,20}$	$c+t+\alpha j+\varepsilon m \quad (\alpha\neq\pm 1)$	$r_{3,11}$
$r^{\varepsilon,\alpha}_{1,21}$	$d+\alpha j+\varepsilon m \quad (\alpha\geq 0)$	$r_{3,12}$
$r^{\alpha,\beta}_{1,22}$	$s+\alpha j+\beta(c+t) \quad (\beta\neq 0)$	$r_{3,13}$
$r^{\alpha,\beta}_{1,23}$	$s+\alpha j+\beta d \quad (\beta>0),(\beta\neq 1 \text{ if } \alpha=0)$	$r_{3,14}$

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The gradient formula for the O(5)⊃O(3) chain of groups

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It is well known how to expand in spherical harmonics the gradient of a radial function in turn multiplied by a spherical harmonic. This expansion involves the use of the Wigner–Eckart theorem for the familiar O(3)⊃O(2) chain of groups, and leads to Wigner coefficients in the formula together with reduced matrix elements which are simple first order differential operators in the radial variable. In the present paper we extend the above analysis to the application of the momentum operator π_m to functions of the collective coordinates α_m, $m = 2,1,0,-1,-2$ associated with quadrupole vibrations. The spherical harmonics are now replaced by the complete but nonorthonormal set of functions χ^{λ}_{sLM}, characterized by the irreducible representations λ, L, M of the O(5)⊃O(3)⊃O(2) chain of groups as well as by an extra labelling index s, that were derived in a previous publication. The application of the gradient to a product of a function $F(\beta)$, $\beta^2 = \Sigma_m \alpha_m \alpha^m$, by χ^{λ}_{sLM} requires an extension of the Wigner–Eckart theorem for the nonorthonormal basis. Results similar to the ones mentioned in the previous paragraph are obtained, though, of course, now we will have Wigner coefficients in the O(5)⊃O(3)⊃O(2) chain which have already been derived and programmed. With the help of the gradient formula we discuss the effect of the operators $[\pi\times\pi]^L_m$, $L = 0,2,4$, $[\alpha\times\pi]^L_m$, L = 1,3 on basis of the O(5)⊃O(3) chain of groups and indicate some of their applications.

1. INTRODUCTION

In recent publications[1,2] a systematic analysis was given of the group theory underlying the collective model of the nucleus introduced originally by Bohr and Mottelson.[3] As is well known this model was fundamental in the understanding of many features of nuclear structure.

The chain of groups involved is U(5)⊃O(5)⊃O(3) as the basic problem is a five-dimensional oscillator related with the quadrupole vibrations of the nucleus, whose states are characterized by a definite angular momentum. These states were explicitly determined in Ref. 2, and with the help of them the concept of reduced $3j$-symbol in the O(5)⊃O(3) chain of groups was defined and then computed numerically for some cases of interest.[4]

As shown in Ref. 2, all matrix elements of polynomial functions of the collective coordinates α_m, $m = 2, 1, 0, -1, -2$, with respect to the states mentioned above, can be obtained with the help of the reduced $3j$-symbol in the O(5)⊃O(3) chain. There remains, though, the question of how to calculate matrix elements of polynomials in the momentum $\pi^{m'}$ conjugate to α_m, i.e.,

$$[\alpha_m, \pi^{m'}] = i\delta_m^{m'}, \quad \pi_m = (-1)^m \pi^{-m} = -i(-1)^m \partial/\partial\alpha_{-m}. \tag{1.1}$$

Of course, the existence of the reduced $3j$-symbol just mentioned suggests the use of the Wigner–Eckart theorem as applied to the O(5)⊃O(3) chain of groups. Some caution has to be used from the fact that our basic states, though complete and characterized by irreducible representations λ of O(5), L of O(3), and M of O(2), have still another label not associated with a Hermitian operator and therefore are not orthonormal. Thus for the expansions we require not only these states but also those dual to them whose scalar product with the original ones gives Kronecker deltas in all the indices involved. This leads to the concept of Wigner coefficient in the O(5)⊃O(3) chain rather than the reduced $3j$-symbol, where the latter, of course, has more symmetry properties.

We shall show that with the help of the Wigner coefficients and their appropriately defined duals, the derivation of the matrix elements of π_m, as well as of some simple functions of them, is straightforward.

To make the method more transport, we shall start by implementing it in the O(3)⊃O(2) chain of groups in which one gets the familiar gradient formula of Rose's book.[5]

The explicit calculation of the matrix elements of the operators $[\pi\times\pi]^L_M$, $L = 0, 2, 4$, are relevant to the evaluation of eigenstates and eigenvalues of the collective Hamiltonians $H(\alpha_m, \pi^{m'})$ proposed recently by Greiner and his collaborators.[6] Furthermore, these operators, together with $[\alpha\times\pi]^L_M$, $L = 1, 3$, will allow us to discuss the effect of the generators of U(5) on states characterized by the irreducible representation of the O(5)⊃O(3) chain of groups. These are some of the motivations for the present analysis.

2. THE GRADIENT FORMULA FOR THE O(3)⊃O(2) CHAIN

For the O(3)⊃O(2) chain the states that are basis for irreducible representations are the spherical harmonics $Y_{lm}(\theta,\varphi)$, and the gradient formula refers to the determination of

$$p_m f(r) Y_{l'm'}(\theta,\varphi), \tag{2.1a}$$

$$p_m = -i(-1)^m \partial/\partial X_{-m}, \tag{2.1b}$$

(where $m' = l', \ldots, -l'$, $m = 1, 0, -1$) as a combination

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of spherical harmonics multiplied by a first order differential operator in r acting on $f(r)$.[5]

To begin with, we note that the integral over three spherical harmonics

$$\{lm, l'm' | l''m''\}$$
$$\equiv \int_0^{\pi}\int_0^{2\pi} Y^*_{l''m''}(\theta,\varphi)\, Y_{lm}(\theta,\varphi)\, Y_{l'm'}(\theta,\varphi)\sin\theta\, d\theta\, d\varphi$$
$$= \left[\frac{(2l+1)(2l'+1)}{4\pi(2l''+1)}\right]^{1/2} \langle l0, l'0 | l''0\rangle \langle lm, l'm' | l''m''\rangle \tag{2.2}$$

carries already the same dependence on the indices m, m', m'' as the Clebsch–Gordan coefficients $\langle lm, l'm' | l''m''\rangle$, because the other factors in (2.2) are independent of them. It is important to stress, though, that the coefficients (2.2) vanish when $l+l'+l''$ is odd as in that case[5] $\langle l0, l'0 | l''0\rangle = 0$. Thus we can replace $\langle lm, l'm' | l''m''\rangle$ by (2.2) only if $l+l'+l''$ is even. We note, though, that the momentum p_m of (2.1b) has the same properties as the position operator x_m, i.e., both of them are polar vectors and thus p_m transforms under the full group O(3), including inversions, as $Y_{1m}(\theta,\varphi)$. Thus in the application of p_m to some function of r multiplied by $Y_{l'm'}(\theta,\varphi)$, we can use the Wigner–Eckart theorem where the usual Wigner coefficients are replaced by (2.2), i.e.,

$$p_m f(r)\, Y_{l'm'}(\theta,\varphi)$$
$$= \sum_{l''m''} Y_{l''m''}(\theta,\varphi) \int_0^{\pi}\int_0^{2\pi} Y^*_{l''m''}[p_m f(r)\, Y_{l'm'}]\sin\theta\, d\theta\, d\varphi$$
$$= \sum_{l''m''} Y_{l''m''}(\theta,\varphi)[\langle l'' \| p \| l'\rangle f(r)]\{1m, l'm' | l''m''\}, \tag{2.3}$$

where all the dependence on the indices m, m', m'' is carried by the coefficient $\{1m, l'm' | l''m''\}$. The reduced matrix element $\langle l'' \| p \| l'\rangle$ depends only on l', l'' and, from the differential form (2.1b) of the operator p_m, it must be a function of r, d/dr of first order in the latter.

To determine this reduced matrix element, let us consider the particular case when $m=-1$, $m'=l'$, i.e.,

$$p_{-1} f(r)\, Y_{l'l'}(\theta,\varphi)$$
$$= i\frac{\partial}{\partial x_1}\left\{ f(r)\left[\frac{(2l'+1)!!}{4\pi l'!}\right]^{1/2}\left(\frac{x_1}{r}\right)^{l'}\right\}$$
$$= -i\left\{\left(r^{l'}\frac{d}{dr}r^{-l'}f\right)\frac{x_{-1}}{r}\left[\frac{2l'+1)!!}{4\pi l'!}\right]^{1/2}\left(\frac{x_1}{r}\right)^{l'}\right.$$
$$\left. -\frac{l'f}{r}\left[\frac{2l'+1}{l'}\right]^{1/2}\left[\frac{(2l'-1)!!}{4\pi(l'-1)!}\right]^{1/2}\left(\frac{x_1}{r}\right)^{l'-1}\right\}. \tag{2.4}$$

We note, though, from the explicit form of Y_{ll} given in (2.4) and from the fact that $x_{-1}/r = (4\pi/3)^{1/2} Y_{1-1}(\theta,\varphi)$ that we may write

$$p_{-1} f(r) Y_{l'l'}(\theta,\varphi)$$
$$= -i\left\{\left[\frac{d}{dr}-\frac{l'}{r}\right] f \times \sum_{l''}\left[\left(\frac{4\pi}{3}\right)^{1/2} \times \{1,-1; l'l' | l'', l'-1\} Y_{l'',l'-1}(\theta,\varphi)\right]\right.$$
$$\left. - [l'(2l'+1)]^{1/2}(f/r) Y_{l'-1,l'-1}(\theta,\varphi)\right\}. \tag{2.5}$$

Before comparing (2.5) with (2.3) to determine the reduced matrix elements, we note that l'' is restricted, by the laws of composition of representations of the O(3) group, to $|l'-1| \le l'' \le l'+1$ and that, furthermore, parity considerations mentioned above forbid $l''=l'$. Thus we have $l''=l'\pm 1$, and for the case $l''=l'+1$ we immediately obtain

$$\langle l'+1 \| p \| l'\rangle = \left(\frac{4\pi}{3}\right)^{1/2}\frac{1}{i}\left(\frac{d}{dr}-\frac{l'}{r}\right). \tag{2.6a}$$

For the case $l''=l'-1$ we require the explicit value of

$$\{1,-1; l', l' | l'-1, l'-1\}$$
$$= [-\int_0^{\pi}\int_0^{2\pi} Y^*_{l'l'}(\theta,\varphi)\; Y_{11}(\theta,\varphi)\; Y_{l'-1,l'-1}(\theta,\varphi)\sin\theta\, d\theta\, d\varphi]^*, \tag{2.7}$$

where we used the relation $Y_{1,-1} = -Y^*_{11}$.

The last expression is very easy to obtain from the explicit form of $Y_{l'l'}$ appearing in (2.4), and thus we have

$$\{1,-1; l', l' | l'-1, l'-1\} = -\left[\frac{3}{4\pi}\,\frac{l'}{2l'+1}\right]^{1/2}. \tag{2.8}$$

From this result we see, when comparing the coefficients of $Y_{l'-1,l'-1}$ in (2.3) and (2.5), that

$$\langle l'-1 \| p \| l'\rangle = \left(\frac{4\pi}{3}\right)^{1/2}\frac{1}{i}\left(\frac{d}{dr}+\frac{l'+1}{r}\right). \tag{2.6b}$$

We did this elementary analysis in such detail because we want to follow it step by step in the derivation of the gradient formula for the O(5)⊃O(3) chain of groups. As a first consideration in the implementation of this program, we discuss in the next section the basis for irreducible representations of the O(5)⊃O(3) chain of groups which are the equivalent ones for this problem to the spherical harmonics for O(3)⊃O(2).

The basis mentioned was derived explicitly in Ref. 2, but besides establishing its properties and notation we shall also discuss one dual to it which shall prove useful in the derivation of the new gradient formula.

3. BASIS FOR IRREDUCIBLE REPRESENTATIONS OF THE O(5) ⊃ O(3) CHAIN OF GROUPS

In Ref. 2 we obtained explicitly the polynomials $P_{\lambda\mu L}(\alpha_m)$ of the collective coordinates α_m, $m = 2, 1, 0, -1, -2$ which are eigenstates of the Casimir operators Λ^2, L^2 of the O(5), O(3) groups with eigenvalues $\lambda(\lambda+3)$, $L(L+1)$. These polynomials have maximum projection in the angular momentum, i.e., $M=L$ and besides are also characterized by a missing label index μ which is restricted to nonnegative integer values in the range[1,2]

$$\lambda - L \leq 3\mu \leq \lambda - (L/2), \qquad \text{if } L \text{ even}, \tag{3.1a}$$

$$\lambda - L \leq 3\mu \leq \lambda - \left(\frac{L+3}{2}\right), \quad \text{if } L \text{ odd}. \tag{3.1b}$$

For a fixed λ, L the relations (3.1) indicate that if there are μ's that satisfy them, they will take all possible integer values between a minimum one μ_0 and a maximum one $\bar{\mu}_0 \geq \mu_0$. Thus instead of the index μ we could use an index s defined by

$$s \equiv \mu - \mu_0 + 1, \tag{3.2a}$$

where

$$s = 1, 2, \ldots, d(\lambda, L), \quad d(\lambda, L) = \bar{\mu}_0 - \mu_0 + 1, \tag{3.2b}$$

with $d(\lambda, L)$ being the number of irreducible representations L of O(3) contained in a given irreducible representation λ of O(5).

As indicated in Ref. 2, we can pass from the collective coordinates α_m in the reference frame fixed in space to those a_m fixed in the body through the transformation

$$\alpha_m = \sum_{m'} D^{2*}_{mm'}(\vartheta_i)\, a_{m'}, \tag{3.3a}$$

where[3]

$$a_2 = a_{-2} = (1/\sqrt{2})\,\beta \sin\gamma, \quad a_1 = a_{-1} = 0, \quad a_0 = \beta\cos\gamma. \tag{3.3b}$$

In (3.3) ϑ_i, $i = 1, 2, 3$, are the Euler angles, β, γ the remaining coordinates, and $D^l_{mm'}(\vartheta_i)$ the Wigner function associated with the irreducible representations of O(3). In these new coordinates the polynomial $P_{\lambda\mu L}(\alpha_m)$ takes the form

$$P_{\lambda\mu L}(\alpha_m) = \beta^\lambda \sum_K \phi_K^{\lambda\mu L}(\gamma)\, D^{L*}_{LK}(\vartheta_i), \tag{3.4}$$

where again the functions $\phi_K^{\lambda\mu L}(\gamma)$ were explicitly given in Ref. 2. Clearly states of arbitrary projection M of the angular momentum could be obtained just by replacing in (3.4) the lower index L in $D^{L*}_{LK}(\vartheta_i)$ by M.

As $\beta^2 = \sum_m \alpha_m \alpha^m$ is an invariant not only of O(3) but also of O(5), it is clear that a complete, though not orthonormal, basis of the O(5)⊃O(3)⊃O(2) chain of groups, corresponding to the $Y_{lm}(\theta, \varphi)$ in the O(3)⊃O(2) chain, is given by

$$\chi^\lambda_{sLM}(\gamma, \vartheta_i) = \pi^{5/4}\, 2^{-\lambda/2} \sum_K \phi_K^{\lambda\mu L}(\gamma)\, D^{L*}_{MK}(\vartheta_i). \tag{3.5}$$

The relation between s and μ is given in (3.2) and the factor $\pi^{5/4} 2^{-\lambda/2}$ with $\pi = 3.1416$ is introduced to eliminate irrelevant terms of the same form in the definition of $P_{\lambda\mu L}(\alpha_m)$.

Besides the complete set of states (3.5) we shall require a dual one which is orthonormal to it in *all* the indices. For this purpose let us first note that as χ^λ_{sLM} is an eigenstate of the Hermitian operators Λ^2, L^2, L_z, we get

$$\int \chi^{\lambda*}_{sLM}(\gamma, \vartheta_i)\, \chi^{\lambda'}_{s'L'M'}(\gamma, \vartheta_i)\, d\tau$$
$$= \delta_{\lambda\lambda'}\delta_{LL'}\delta_{MM'}\, 2^{3-\lambda}\pi^{9/2}(2L+1)^{-1}$$
$$\times \int_0^\pi \sum_K \phi_K^{\lambda\mu L*}(\gamma)\, \phi_K^{\lambda\mu' L}(\gamma) \sin 3\gamma\, d\gamma$$
$$\equiv \delta_{\lambda\lambda'}\delta_{LL'}\delta_{MM'} M_{ss'}(\lambda, L), \tag{3.6a}$$

where in (3.6a) the volume element $d\tau$ is

$$d\tau = \sin 3\gamma\, d\gamma \sin\vartheta_2\, d\vartheta_1\, d\vartheta_2\, d\vartheta_3, \tag{3.6b}$$

and the ranges of integration of the angles are given by

$$0 \leq \vartheta_2, \quad \gamma \leq \pi, \quad 0 \leq \vartheta_1, \vartheta_3 \leq 2\pi, \tag{3.6c}$$

where in formula (3.6) [and also (3.11), (3.13)] the π is 3.1416 and *not* the momentum operator.

The coefficients $M_{ss'}(\lambda, L)$ have been calculated numerically as they are in fact particular reduced $3j$-symbols in the O(5)⊃O(3) chain.[4]

The $d(\lambda, L) \times d(\lambda, L)$ Hermitian matrix

$$M(\lambda, L) = \|M_{ss'}(\lambda, L)\| \tag{3.7}$$

can be inverted, and we shall denote the components of the latter as $M^{-1}_{ss'}(\lambda, L)$.

We proceed to denote the set of states dual to the $\chi^\lambda_{sLM}(\gamma, \vartheta_i)$ of (3.5) by the same symbol but with a bar above it and define them as

$$\bar{\chi}_{sLM}(\gamma, \vartheta_i) = \sum_{\bar{s}} M^{-1}_{s\bar{s}}(\lambda, L)\, \chi^\lambda_{\bar{s}LM}(\gamma, \vartheta_i). \tag{3.8}$$

Clearly we have then that

$$\int \bar{\chi}^{\lambda*}_{sLM}(\gamma, \vartheta_i)\, \chi^{\lambda'}_{s'L'M'}(\gamma, \vartheta_i)\, d\tau = \delta_{\lambda\lambda'}\delta_{ss'}\delta_{LL'}\delta_{MM'}. \tag{3.9}$$

With the help of the relation (3.9) we can immediately find out the coefficients in the expansion of an arbitrary function of the γ, ϑ_i in terms of the $\chi^\lambda_{sLM}(\gamma, \vartheta_i)$. In particular we see that a product of two functions of the type (3.5) can be expanded as

$$\chi^\lambda_{sLM}(\gamma, \vartheta_i)\, \chi^{\lambda'}_{s'L'M'}(\gamma, \vartheta_i)$$
$$= \sum_{\lambda''} \sum_{s''L''M''} \{\lambda sLM, \lambda's'L'M' \,|\, \lambda''s''L''M''\}\, \chi^{\lambda''}_{s''L''M''}(\gamma, \vartheta_i), \tag{3.10}$$

where

$$\{\lambda sLM, \lambda's'L'M' \,|\, \lambda''s''L''M''\}$$
$$\equiv \int \bar{\chi}^{\lambda''*}_{s''L''M''}(\gamma, \vartheta_i)\, \chi^\lambda_{sLM}(\gamma, \vartheta_i)\, \chi^{\lambda'}_{s'L'M'}(\gamma, \vartheta_i)\, d\tau$$
$$= \pi^{23/4}\, 2^{(6-\lambda-\lambda'-\lambda'')/2} \langle LM, L'M' \,|\, L''M''\rangle$$
$$\times (2L''+1)^{-1/2}(-1)^{L+L'+L''} \sum_{\bar{s}''} M^{-1}_{s''\bar{s}''}(\lambda'', L'')$$
$$\times (\lambda, s+\mu_0-1, L; \lambda', s'+\mu_0'-1, L'; \lambda'', \bar{s}''+\mu_0''-1, L''), \tag{3.11}$$

and in turn the last parenthesis in (3.11) is the reduced $3j$-symbol in the O(5)⊃O(3) chain of groups defined by[2]

$$(\lambda\mu L, \lambda'\mu'L', \lambda''\mu''L'')$$
$$\equiv \int_0^\pi \sum_{K, K', K''} \begin{pmatrix} L & L' & L'' \\ K & K' & K'' \end{pmatrix} \phi_K^{\lambda\mu L}(\gamma)\, \phi_{K'}^{\lambda'\mu'L'}(\gamma)$$
$$\times \phi_{K''}^{\lambda''\mu''L''}(\gamma) \sin 3\gamma\, d\gamma, \tag{3.12}$$

where, using (3.2a), we have replaced the μ's by $s + \mu_0 - 1$.

The coefficient (3.11) will be called the Wigner coefficient for the O(5)⊃O(3) chain of groups, as its definition parallels the one given by (2.2) for the O(3)⊃O(2) chain.

In view of the fact that certain important cases of (3.12) have already been programmed numerically,[4] we see from (3.11) that the corresponding Wigner coefficients are also available. These coefficients, together with their duals defined by

$$\overline{\{\lambda s L M, \lambda' s' L' M' | \lambda'' s'' L'' M''\}}$$
$$\equiv \int \chi^{\lambda''*}_{s''L''M''}(\gamma, \vartheta_i)\, \overline{\chi}^{\lambda}_{sLM}(\gamma, \vartheta_i)\, \overline{\chi}^{\lambda'}_{s'L'M'}(\gamma, \vartheta_i)\, d\tau$$
$$= \pi^{23/4}\, 2^{(6-\lambda-\lambda'-\lambda'')/2} \langle LM, L'M' | L''M''\rangle$$
$$\times (2L''+1)^{-1/2}(-1)^{L+L'+L''} \sum_{\bar{s}\bar{s}'} M^{-1}_{s\bar{s}}(\lambda, L)\, M^{-1}_{s'\bar{s}'}(\lambda', L')$$
$$\times (\lambda, \bar{s}+\mu_0-1, L; \lambda', \bar{s}'+\mu_0'-1, L'; \lambda'', s''+\mu_0''-1, L'') \tag{3.13}$$

will be required in the derivation of the gradient formula for the O(5)⊃O(3) chain of groups and in the composition of products of states corresponding to given irreducible representations of the same chain.

4. THE GRADIENT FORMULA

We are now interested in developing

$$\pi_m F(\beta)\, \chi^{\lambda'}_{s'L'M'}(\gamma, \vartheta_i), \tag{4.1}$$

where π_m is the operator (1.1), in terms of the χ^{λ}_{sLM} of (3.5) with coefficients given by first order differential operators in β acting on $F(\beta)$. To achieve this objective, we shall make use of the Wigner—Eckart theorem for states characterized by irreducible representations of the O(5)⊃O(3) chain of groups, a theorem whose validity for the states in question we discuss in the Appendix.

We first prove that α_m, π_m transform exactly in the same way under arbitrary transformations of O(5). Assuming a linear transformation

$$\alpha'_{m'} = \sum_m R^m_{m'} \alpha_m, \tag{4.2}$$

subject to the condition that

$$\sum_{m'} \alpha'^{m'} \alpha'_{m'} = \sum_m \alpha^m \alpha_m, \tag{4.3}$$

we immediately obtain that

$$\alpha^m = \sum_{m'} R^m_{m'} \alpha'^{m'}, \tag{4.4}$$

and thus we have

$$R^m_{m'} = \frac{\partial \alpha'_{m'}}{\partial \alpha_m} = \frac{\partial \alpha^m}{\partial \alpha'^{m'}}, \tag{4.5}$$

where covariant and contravariant components are related by

$$\alpha^m = (-1)^m \alpha_{-m}. \tag{4.6}$$

As

$$\pi'_{m'} = \frac{1}{i}\frac{\partial}{\partial \alpha'^{m'}} = \frac{1}{i}\sum_m \frac{\partial \alpha^m}{\partial \alpha'^{m'}}\frac{\partial}{\partial \alpha^m} = \sum_m R^m_{m'} \pi_m, \tag{4.7}$$

we conclude that π_m transforms exactly as

$$\alpha_m = \beta \chi^1_{12m}(\gamma, \vartheta_i), \quad m = 2, 1, 0, -1, -2, \tag{4.8}$$

where the last relation follows from the definitions (3.5), (3.1).

In analogy to the gradient formulas (2.3) for the O(3)⊃O(2) chain, we conclude from (3.8), (3.9), (3.10), that we may now write

$$\pi_m F(\beta)\, \chi^{\lambda'}_{s'L'M'}(\gamma, \vartheta_i)$$
$$= \sum_{\lambda''} \sum_{s''L''M''} \chi^{\lambda''}_{s''L''M''}(\gamma, \vartheta_i) \int \overline{\chi}^{\lambda''*}_{s''L''M''}[\pi_m F(\beta)\, \chi^{\lambda'}_{s'L'M'}]\, d\tau$$
$$= \sum_{\lambda''} \sum_{s''L''M''} \chi^{\lambda''}_{s''L''M''}(\gamma, \vartheta_i)[\langle \lambda'' \| \pi \| \lambda' \rangle F(\beta)]$$
$$\times \{112m; \lambda' s' L' M' | \lambda'' s'' L'' M''\}, \tag{4.9}$$

where the reduced matrix element $\langle \lambda'' \| \pi \| \lambda' \rangle$ is a first order differential operator in β that depends on the indices λ', λ'' of the irreducible representations of O(5) but on none of the others.

As in section 2 for the O(3)⊃O(2) chain, we shall determine the reduced matrix elements through the analysis of a particular case. We shall consider in a given representation λ' of O(5) the state with highest possible L' and $M' = L'$. From the inequalities (3.1) we conclude that this value is $L' = 2\lambda'$ and for it $\mu' = 0$ so that from (3.2) $s' = 1$. From the discussion in Ref. 2 this state has the form

$$\chi^{\lambda'}_{1,2\lambda',2\lambda'}(\gamma, \vartheta_i) = \left(\frac{\alpha_2}{\beta}\right)^{\lambda'}, \tag{4.10}$$

and thus if we apply to it π_m with $m = -2$ we get

$$\pi_{-2}\left[F(\beta)\left(\frac{\alpha_2}{\beta}\right)^{\lambda'}\right] = -i\frac{\partial}{\partial \alpha_2}\left[\left(\frac{F(\beta)}{\beta^{\lambda'}}\right)\alpha_2^{\lambda'}\right]$$
$$= -i\left[\left(\beta^{\lambda'}\frac{d}{d\beta}\beta^{-\lambda'}F\right)\frac{\alpha_{-2}}{\beta}\left(\frac{\alpha_2}{\beta}\right)^{\lambda'} + \frac{\lambda' F}{\beta}\left(\frac{\alpha_2}{\beta}\right)^{\lambda'-1}\right]. \tag{4.11}$$

We note though from (4.8), (4.10) and the expansion (3.10) that the expression (4.11) can be written as

$$\pi_{-2} F(\beta)\, \chi^{\lambda'}_{1,2\lambda',2\lambda'}(\gamma, \vartheta_i)$$
$$= -i\left\{\left[\left(\frac{d}{d\beta} - \frac{\lambda'}{\beta}\right)F(\beta)\right] \sum_{\lambda''} \sum_{s''L''M''}\left[\chi^{\lambda''}_{s''L''M''}(\gamma, \vartheta_i)\right.\right.$$
$$\left.\times \{1, 1, 2, -2; \lambda', 1, 2\lambda', 2\lambda' | \lambda'' s'' L'' M''\}\right]$$
$$\left. + \frac{\lambda' F(\beta)}{\beta}\chi^{\lambda'-1}_{1,2\lambda'-2,2\lambda'-2}(\gamma, \vartheta_i)\right\}. \tag{4.12}$$

Before comparing (4.12) with (4.9) to determine the reduced matrix element, we note that in the Wigner coefficient appearing in them the irreducible representations λ' and 1 of O(5) are combined to give λ''. The selection rules[7] indicate then that $\lambda'' = \lambda' \pm 1$ if, as required in the present case, the resulting representation is symmetric. For $\lambda'' = \lambda' + 1$ we immediately obtain from comparing (4.9), where we take $s' = 1$, $L' = M' = 2\lambda'$, with (4.12), that

$$\langle \lambda' + 1 \| \pi \| \lambda' \rangle = \frac{1}{i}\left(\frac{d}{d\beta} - \frac{\lambda'}{\beta}\right). \tag{4.13a}$$

For $\lambda'' = \lambda' - 1$ we require the explicit value of

$$Q(\lambda')\equiv\{1,1,2,-2;\lambda',1,2\lambda',2\lambda'|\lambda'-1,1,2\lambda'-2,2\lambda'-2\}$$
$$=M_{11}^{-1}(\lambda'-1,2\lambda'-2)\int\chi^{\lambda'-1*}_{1,2\lambda'-2,2\lambda'-2}\chi^{1}_{1,2,-2}\chi^{\lambda'}_{1,2\lambda',2\lambda'}\,d\tau$$
$$=M_{11}^{-1}(\lambda'-1,2\lambda'-2)[\Gamma(\lambda'+\tfrac{5}{2})/2]^{-1}$$
$$\times[\int(\alpha_2^{\lambda'})^*\alpha_2\alpha_2^{\lambda'-1}\,d^5\alpha]^*, \qquad (4.14)$$

where in the last parenthesis of (4.14) we extended the integration to the full volume element of the α_m variables by multiplying and dividing the Wigner coefficient by

$$\int_0^\infty\beta^{2\lambda'}\exp(-\beta^2)\,\beta^4\,d\beta=\Gamma(\lambda'+\tfrac{5}{2})/2. \qquad (4.15)$$

The last integral in (4.14) is then trivial to evaluate and a similar trick can be used to calculate $M_{11}(\lambda'-1, 2\lambda'-2)$ to obtain finally

$$Q(\lambda')=\lambda'(2\lambda'+3)^{-1}. \qquad (4.16)$$

Comparing then the coefficients of $\chi^{\lambda'-1}_{1,2\lambda'-2,2\lambda'-2}$ in (4.9), (4.12), we then conclude that

$$\langle\lambda'-1\|\pi\|\lambda'\rangle=\frac{1}{i}\left(\frac{d}{d\beta}+\frac{\lambda'+3}{\beta}\right). \qquad (4.13b)$$

The gradient formula for the O(5)$\supset$O(3) chain of groups is then given by (4.9) in which $\lambda''=\lambda'\pm1$ and the reduced matrix elements have the operator form (4.13). In the next section we shall proceed to indicate its usefulness in the analysis of matrix elements of velocity dependent collective Hamiltonians.

5. MATRIX ELEMENTS OF VELOCITY DEPENDENT COLLECTIVE HAMILTONIANS

In the extensive work of Greiner and his collaborators[6] they used Hamiltonians of the type $H(\alpha_m,\pi^{m'})$, which are, of course, invariant under O(3). The momenta π_m appear in them at most to the second order so that using the commutation rules (1.1) we have either terms containing only the α_m's alone, whose matrix elements we already discussed in Ref. 2, or terms of the form

$$\sum_{\lambda msL}B_{\lambda msL}\,\beta^{2m+\lambda}\sum_M(-1)^M\chi^{\lambda}_{s,L,-M}(\gamma,\vartheta_i)[\pi\times\pi]^L_M, \qquad (5.1)$$

where B are arbitrary coefficients. Linear terms in π cannot remain as the corresponding Hamiltonian would not be invariant under time reflection.

By introducing intermediate states characterized by $\lambda's'L'M'$ between the χ functions and the momentum dependent part appearing in (5.1) we can reduce the evaluation of the matrix elements to Wigner coefficients of the type (3.11) multiplied by the matrix elements of $[\pi\times\pi]^L_M$, which we proceed to analyze here.

From the discussion of the previous section we can write

$$[\pi\times\pi]^L_M F(\beta)\chi^{\lambda'}_{s'L'M'}(\gamma,\vartheta_i)$$
$$=\sum_{\lambda''s''L''M''}\sum_{\lambda'''s'''L'''M'''}\sum_{m,m'}\chi^{\lambda''}_{s''L''M''}(\gamma,\vartheta_i)\langle 2m,2m'|LM\rangle$$
$$\times[\{1,1,2,m;\lambda'''s'''L'''M'''|\lambda''s''L''M''\}\langle\lambda''\|\pi\|\lambda'''\rangle$$
$$\times\{1,1,2,m';\lambda's'L'M'|\lambda'''s'''L'''M'''\}$$
$$\times\langle\lambda'''\|\pi\|\lambda'\rangle F(\beta)], \qquad (5.2)$$

where the reduced matrix element operators $\langle\lambda''\|\pi\|\lambda'\rangle$ are given by (4.13) and the Wigner coefficients in the O(5)$\supset$O(3) chain of groups have the form (3.11).

Clearly L in (5.2) is restricted to $L=0,2,4$. The case $L=0$ is trivial as $[\pi\times\pi]^0_0$ is proportional to the Laplacian which can be expressed in terms of a second order differential operator in β and the Casimir operator Λ^2 of O(5) for which the χ^{λ}_{sLM} are eigenstates with eigenvalue $\lambda(\lambda+3)$. Thus we need to consider only the case $[\pi\times\pi]^L_M$ where $L=2,4$, which, from the transformation properties (4.7) of the π_m and (4.2) of the α_m, transform in exactly the same way under O(5) as $[\alpha\times\alpha]^L_M$, $L=2,4$.

We note, though, from the definition (3.5) of χ^{λ}_{sLM} and the explicit form of the polynomials $P_{\lambda\mu L}(\alpha_m)$ given in Ref. 2, that

$$\chi^2_{1LM}(\gamma,\vartheta_i)=c_L\beta^{-2}[\alpha\times\alpha]^L_M,\quad L=2,4, \qquad (5.3a)$$

where

$$c_2=3\sqrt{7},\quad c_4=1. \qquad (5.3b)$$

Thus we conclude that

$$c_L[\pi\times\pi]^L_M,\quad L=2,4, \qquad (5.4)$$

transforms in the same way under O(5) as χ^2_{1LM} so that we expect the appearance in (5.2) of the Wigner coefficients

$$\{21LM;\lambda's'L'M'|\lambda''s''L''M''\}$$
$$=\int\bar{\chi}^{\lambda''*}_{s''L''M''}c_L\beta^{-2}[\alpha\times\alpha]^L_M\chi^{\lambda'}_{s'L'M'}\,d\tau$$
$$=c_L\sum_{m,m'}\langle 2m,2m'|LM\rangle$$
$$\times\sum_{\lambda'''s'''L'''M'''}[\{112m;\lambda'''s'''L'''M'''|\lambda''s''L''M''\}$$
$$\times\{112m';\lambda's'L'M'|\lambda'''s'''L'''M'''\}], \qquad (5.5)$$

where we made use of the fact that from (4.8)

$$\beta^{-2}[\alpha\times\alpha]^L_M$$
$$=\sum_{m,m'}\langle 2m,2m'|LM\rangle\chi^1_{12m}(\gamma,\vartheta_i)\chi^1_{12m'}(\gamma,\vartheta_i), \qquad (5.6)$$

and we introduced an intermediate state $\chi^{\lambda'''}_{s'''L'''M'''}$ between the χ's appearing in (5.5), (5.6).

To see that in fact the Wigner coefficients (5.5) are present in (5.2) we start by noting that in these equations $\lambda'''=\lambda'\pm1$ and $\lambda''=\lambda'''\pm1$ so that $\lambda''=\lambda'+2$, λ', $\lambda'-2$. In the case $\lambda''=\lambda'+2$ we can have only one intermediate state $\lambda'''=\lambda'+1$ both in (5.2) and (5.5) and thus there is no summation over the intermediate λ''' though there may still remain one over $s'''L'''M'''$. Thus clearly the Wigner coefficient (5.5) appears in (5.2) for the case $\lambda''=\lambda'+2$ multiplied by the reduced matrix operator

$$\langle\lambda'+2\|[\pi\times\pi]\|\lambda'\rangle$$
$$\equiv\langle\lambda'+2\|\pi\|\lambda'+1\rangle\langle\lambda'+1\|\pi\|\lambda'\rangle$$
$$=-\left(\frac{d}{d\beta}-\frac{\lambda'+1}{\beta}\right)\left(\frac{d}{d\beta}-\frac{\lambda'}{\beta}\right). \qquad (5.7a)$$

A similar consideration holds for $\lambda''=\lambda'-2$ where there is only a single $\lambda'''=\lambda'-1$ and thus the Wigner coefficient (5.5) appears in (5.2) multiplied by the reduced matrix operator

$$\langle\lambda'-2\|[\pi\times\pi]\|\lambda'\rangle\equiv\langle\lambda'-2\|\pi\|\lambda'-1\rangle\langle\lambda'-1\|\pi\|\lambda'\rangle$$
$$=-\left(\frac{d}{d\beta}+\frac{\lambda'+2}{\beta}\right)\left(\frac{d}{d\beta}+\frac{\lambda'+3}{\beta}\right). \tag{5.7b}$$

There remains the case $\lambda''=\lambda'$ where both $\lambda'''=\lambda'+1$ and $\lambda'''=\lambda'-1$ are possible. We note though that the operator

$$\langle\lambda'\|[\pi\times\pi]\|\lambda'\rangle\equiv\langle\lambda'\|\pi\|\lambda'+1\rangle\langle\lambda'+1\|\pi\|\lambda'\rangle$$
$$=-\left(\frac{1}{\beta^4}\frac{d}{d\beta}\beta^4\frac{d}{d\beta}-\frac{\lambda'(\lambda'+3)}{\beta^2}\right)$$
$$=\langle\lambda'\|\pi\|\lambda'-1\rangle\langle\lambda'-1\|\pi\|\lambda'\rangle \tag{5.7c}$$

has the same value when it is defined with any of the two intermediate $\lambda'''=\lambda'\pm1$. Thus we clearly see that for $\lambda''=\lambda'$ the corresponding Wigner coefficient (5.5) appears multiplied by the reduced matrix element operator (5.7c). Thus we can finally write

$$c_L[\pi\times\pi]^L_M F(\beta)\chi^{\lambda'}_{s'L'M'}(\gamma,\vartheta_i)$$
$$=\sum_{\lambda''}\sum_{s''L''M''}\chi^{\lambda''}_{s''L''M''}(\gamma,\vartheta_i)[\langle\lambda''\|[\pi\times\pi]\|\lambda'\rangle F(\beta)]$$
$$\times\{21LM;\lambda's'L'M'|\lambda''s''L''M''\}. \tag{5.8}$$

We have obtained a formula for $[\pi\times\pi]^L_M$ acting on the basis states of the O(5)⊃O(3) chain of groups similar to the one for π_m on the same basis. We note that the Wigner coefficients appearing in (5.8) are given by (5.5) in terms of

$$\{112m;\lambda's'L'M'|\lambda''s''L''M''\}, \tag{5.9}$$

which are also required in the gradient formula (4.9). From (3.11) these coefficients can be immediately obtained once we have the reduced $3j$-symbol

$$(102;\lambda'\mu'L';\lambda''\mu''L''), \tag{5.10}$$

where s and μ are related by (3.2). This $3j$-symbol has already been programmed by several researchers[4] interested in this field. Thus the matrix elements of Hamiltonians of the type $H(\alpha_m,\pi^{m'})$, where $\pi^{m'}$ is not higher than second order, can be systematically calculated. Furthermore, we plan to program directly the reduced $3j$-symbol

$$(20L;\lambda'\mu'L';\lambda''\mu''L''). \tag{5.11}$$

In fact the analysis of this paper can be extended to problems involving several quadrupole collective coordinates as happens, for example, when we consider the interaction between two even—even nuclei, each one with its own collective coordinates $\alpha^{(1)}_m$, $\alpha^{(2)}_m$, $m=2,1,0,-1,-2$. We require then the construction, from products of states in these coordinates characterized by given irreducible representations of the O(5)⊃O(3) chain of groups, of composite states characterized in the same fashion. We show in the Appendix that this objective can be achieved with the help of the dual of the Wigner coefficient defined by (3.13). Once these composite states are available, the determination of matrix elements of some potential interaction such as $V(\alpha^{(1)}_m,\alpha^{(2)}_m)$ involve recoupling procedures that lead to concepts such as Racah coefficients for the O(5)⊃O(3) chain of groups.

The observations of the previous paragraph are relevant to some considerations, being developed by the present authors, for the analysis of fission and fusion of heavy nuclei whose internal structure is assumed describable in terms of collective quadrupole excitations.

In the present section we discussed the operator $[\pi\times\pi]^L_M$ because of its relevance for eigenstates and eigenvalues of velocity dependent Hamiltonians $H(\alpha_m,\pi^{m'})$. We shall show in the next section that these operators, together with others of similar type, are also relevant to the application of the generators of U(5) to states characterized by irreducible representations of the O(5)⊃O(3) chain of groups.

6. APPLICATION OF THE GENERATORS OF U(5) TO THE EIGENSTATES IN THE O(5) ⊃ O(3) CHAIN

The generators of U(5) are given in terms of the creation and annihilation operators

$$\eta_m=(1/\sqrt{2})(\alpha_m-i\pi_m),\quad \xi_m=(1/\sqrt{2})(\alpha_m+i\pi_m) \tag{6.1}$$

by the expression[1,2]

$$[\eta\times\xi]^L_M\equiv\sum_{m,m'}\langle 2m,2m'|LM\rangle\eta_m\xi_{m'},\quad L=0,1,2,3,4. \tag{6.2}$$

Clearly we can also write them in terms α_m, π_m, as

$$[\eta\times\xi]^L_M=\tfrac14([\alpha\times\alpha]^L_M+[\pi\times\pi]^L_M)[1+(-1)^L]$$
$$+\tfrac12 i[\alpha\times\pi]^L_M[1-(-1)^L]-(\sqrt5/2)\,\delta_{L0}. \tag{6.3}$$

The application of the operators $[\alpha\times\alpha]^L_M$, $[\pi\times\pi]^L_M$, $L=0,2,4$, to the states

$$F(\beta)\chi^{\lambda'}_{s'L'M'}(\gamma,\vartheta_i) \tag{6.4}$$

was already discussed in the previous section. Thus, there remains only the analysis of $[\alpha\times\pi]^L_M$, $L=1,3$, which is associated with the generators[1,2] of the O(5) subgroup of U(5).

For $L=1$ we have that[1]

$$i[\alpha\times\pi]^1_\mu=(10)^{-1/2}L_\mu,\quad \mu=1,0,-1, \tag{6.5}$$

with L_μ being the components of angular momentum, so that the application of the operator (6.5) to the states (6.4) is trivial. The situation is more interesting for the case $L=3$ where from (4.8), (4.9) we have

$$\int\bar\chi^{\lambda''*}_{s''L''M''}\,i[\alpha\times\pi]^3_M F(\beta)\chi^{\lambda'}_{s'L'M'}\,d\tau$$
$$=i\sum_{\lambda'''}\Big[\sum_{m,m'}\sum_{s'''L'''M'''}\langle 2m,2m'|3M\rangle$$
$$\times\{112m;\lambda'''s'''L'''M'''|\lambda''s''L''M''\}$$
$$\times\{112m';\lambda's'L'M'|\lambda'''s'''L'''M'''\}][\beta\langle\lambda'''\|\pi\|\lambda'\rangle F(\beta)\Big] \tag{6.6}$$

with $\langle\lambda'''\|\pi\|\lambda'\rangle$ given by (4.13). We note though that as $[\alpha\times\pi]^3_M$ is a generator of O(5), it cannot change the index λ' of the original state and thus the formula (6.6) is of interest to us only when $\lambda''=\lambda'$. Furthermore, from (4.13) the formula (6.6) contains a term $\beta d/d\beta$ whose coefficient must be zero as it is obvious that the generators of O(5) are independent of the radial coordinate β. Thus the terms in the first square bracket of the right hand side of (6.6) must be equal and of opposite sign when $\lambda''' = \lambda' + 1$ and $\lambda''' = \lambda' - 1$. This allows us to write, after making use of the expansion (3.11), the following expression for the matrix element of the generators of O(5) in the O(5)$\supset$O(3) basis:

$$\begin{aligned}\int \bar{\chi}^{\lambda' *}_{s''L''M''}[i[\alpha\times\pi]^3_M \chi^{\lambda'}_{s'L'M'}]\,d\tau &= \pi^{23/2}\sqrt{7}\,(2\lambda'+3)\langle L'M', 3M \mid L''M''\rangle (2L''+1)^{-1/2}\,2^{6-2\lambda'} \\ &\times \sum_{s'''L'''} (-1)^{L'''+L'} W(32L''L''';2L') \\ &\sum_{\bar{s}'''\bar{s}''} M^{-1}_{s'''\bar{s}'''}(\lambda'-1, L''') \times M^{-1}_{s''\bar{s}''}(\lambda', L'') \\ &\times (102;\lambda', s'+\mu_0'-1, L'; \lambda'-1, \bar{s}'''+\mu_0'''-1, L''') \\ &\times (102; \lambda'-1, s'''+\mu_0'''-1; \lambda', \bar{s}''+\mu_0''-1, L''),\end{aligned} \tag{6.7}$$

where W is a Racah coefficient[5] and the reduced $3j$-symbols at the end of the formula have already been programmed.[4]

We see then that we can apply the generators (6.2) of U(5) to any state (6.4), which will prove useful in many applications.

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APPENDIX: THE WIGNER-ECKART THEOREM IN THE NONORTHONORMAL BASIS OF THE O(5) ⊃ O(3) CHAIN OF GROUPS

The derivation of the gradient formula and its extensions in Secs. 4 and 5 was carried out in analogy to the discussion in Sec. 2 for the familiar O(3)$\supset$O(2) chain. In the latter one made use of the Wigner–Eckart theorem as discussed, for example, in Rose's[5] or Wigner's[8] books.

To be able to prove a similar theorem for states and tensors characterized by irreducible representations of the O(5)$\supset$O(3) chain of groups, we need to define a complete set of basis states for *all* irreducible representations[7] $\Lambda \equiv (\lambda_1\lambda_2)$ of the O(5) group and not only for the symmetric ones $\Lambda = (\lambda 0)$ that were required in the present paper. Furthermore, on these basis states, in general not orthonormal, and their duals, we define the irreducible representations themselves. With the help of the latter we define the Clebsch–Gordan coefficients and their duals that combine two bases of irreducible representations of independent systems to give new ones that are also irreducible bases with the established row characterization. From them the general properties of representations, valid for all compact groups, allow the immediate derivation of the Wigner–Eckart theorem.

The considerations to be developed below will be entirely abstract, but when restricted to the symmetric representations[7] the basis and the Clebsch–Gordan coefficients mentioned above become the states (3.5) and (except for a factor depending only on the irreducible representations $\lambda, \lambda', \lambda''$) the Wigner coefficients (3.11) and their duals (3.13). Thus the Wigner–Eckart theorem will be proved in the form required in Secs. 4 and 5.

Let us start with the definition of the basis for irreducible representations in the O(5)$\supset$O(3)$\supset$O(2) chain though our notation will be kept general enough so that the results will be valid for any chain of compact groups.

For O(5) the most general representation[7] is characterized by a partition $\Lambda = (\lambda_1, \lambda_2)$, where $\lambda_1 \geq \lambda_2 \geq 0$ and they are integers. The row of the irreducible representation is characterized by the L, M of O(3), O(2) and also by two missing labels[9] t, s which reduce only to the s of (3.2) if the representation is symmetric, i.e., $\Lambda = (\lambda 0)$. We shall use the following compact notation:

$$\rho = sLM, \quad \sigma = t\rho, \quad \Lambda = (\lambda_1, \lambda_2), \tag{A1}$$

and x will be the set of parameters on which the basis depends. If we consider just the symmetric representations $\Lambda = (\lambda 0)$, then

$$x = (\gamma, \vartheta_1, \vartheta_2, \vartheta_3) \tag{A2}$$

or equivalently α_m as $\beta^2 = \sum_m \alpha_m \alpha^m$ is an invariant of O(5). For the general representation $\Lambda = (\lambda_1\lambda_2)$ we will need to associate x with two independent variables $\alpha_m^{(1)}$, $\alpha_m^{(2)}$.

We shall denote our general basis states for irreducible representations of O(5) as

$$\chi^\Lambda_\sigma(x) \tag{A3}$$

and they will clearly be nonorthonormal as t, s are not eigenvalues of Hermitian operators.

We can define a dual basis to the states (A3) by $\bar{\chi}^\Lambda_\sigma$ with the property

$$\int \bar{\chi}^{\Lambda *}_\sigma(x)\, \chi^{\Lambda'}_{\sigma'}(x)\, dx = \delta_{\Lambda\Lambda'}\,\delta_{\sigma\sigma'}\,. \tag{A4}$$

The orthogonal transformation (4.2) of the O(5) group can then be symbolized by

$$x' = Rx, \tag{A5}$$

and following Wigner[8] we define the operator P_R associated with the transformation R and acting on χ^Λ_σ as

$$P_R\,\chi^\Lambda_\sigma(x) = \chi^\Lambda_\sigma(R^{-1}x). \tag{A6}$$

In view of the fact that the states $\chi^\Lambda_\sigma(x)$, for a given Λ and all compatible σ's, form a complete basis for an irreducible representation of O(5), we can expand

$$P_R\,\chi^\Lambda_\sigma(x) = \sum_{\bar{\sigma}} \chi^\Lambda_{\bar{\sigma}}(x)\, \Delta^\Lambda_{\bar{\sigma}\sigma}(R), \tag{A7a}$$

where $\Delta^\Lambda_{\bar{\sigma}\sigma}(R)$ will then be a representation of the O(5) group elements on the basis χ^Λ_σ. We use the notation Δ rather than the familiar D as our basis is not orthonormal and thus the representation is not unitary.

In a similar fashion we have for the dual basis

$$P_R \bar{\chi}_\sigma^\Lambda(x) = \sum_{\bar{\sigma}} \bar{\chi}_{\bar{\sigma}}^\Lambda(x)\, \bar{\Delta}_{\bar{\sigma}\sigma}^\Lambda(R), \tag{A7b}$$

and the two representations are related by

$$\begin{aligned} \Delta_{\bar{\sigma}\sigma}^\Lambda(R) &= \int \bar{\chi}_{\bar{\sigma}}^{\Lambda *}(P_R \chi_\sigma^\Lambda)\, dx \\ &= \int [P_R(P_{R^{-1}}\bar{\chi}_{\bar{\sigma}}^\Lambda)]^*(P_R\chi_\sigma^\Lambda)\, dx \\ &= [\int \chi_\sigma^{\Lambda *}(P_{R^{-1}}\bar{\chi}_{\bar{\sigma}}^\Lambda)\, dx]^* = \bar{\Delta}_{\sigma\bar{\sigma}}^{\Lambda *}(R^{-1}). \end{aligned} \tag{A8}$$

We now proceed to define the Clebsch–Gordan coefficients $\langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda''\sigma''\rangle$ for the O(5) group in an abstract fashion. Take the product of two sets of states $\chi_\sigma^\Lambda(x_1)$, $\chi_{\sigma'}^{\Lambda'}(x_2)$ associated with *independent* variables x_1, x_2 and combine them with coefficients such that the resulting function of x_1, x_2 is a basis for an irreducible representation Λ'' of O(5) with row σ'', i.e.,

$$X_{\sigma''}^{\Lambda\Lambda'\Lambda''}(x_1, x_2) = \sum_{\sigma,\sigma'} \langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda''\sigma''\rangle\, \chi_\sigma^\Lambda(x_1)\, \chi_{\sigma'}^{\Lambda'}(x_2), \tag{A9}$$

where

$$P_R X_{\sigma''}^{\Lambda\Lambda'\Lambda''}(x_1, x_2) = \sum_{\bar{\sigma}''} X_{\bar{\sigma}''}^{\Lambda\Lambda'\Lambda''}(x_1, x_2)\, \Delta_{\bar{\sigma}''\sigma''}^{\Lambda''}(R). \tag{A10}$$

We also define the dual $\bar{X}_{\sigma''}^{\Lambda\Lambda'\Lambda''}(x_1, x_2)$ of the function (A9) by the property

$$\iint \bar{X}_{\sigma'''}^{\Lambda\Lambda'\Lambda'''*}(x_1, x_2)\, X_{\sigma''}^{\Lambda\Lambda'\Lambda''}(x_1, x_2)\, dx_1\, dx_2 = \delta_{\Lambda''\Lambda'''}\, \delta_{\sigma''\sigma'''}, \tag{A11}$$

and the dual Clebsch–Gordan coefficient as

$$\bar{X}_{\sigma'''}^{\Lambda\Lambda'\Lambda'''}(x_1, x_2) = \sum_{\bar{\sigma},\bar{\sigma}'} \overline{\langle\Lambda\bar{\sigma}, \Lambda'\bar{\sigma}' \mid \Lambda'''\sigma'''\rangle}\, \bar{\chi}_{\bar{\sigma}}^\Lambda(x_1)\, \bar{\chi}_{\bar{\sigma}'}^{\Lambda'}(x_2). \tag{A12}$$

Multiplying both sides of (A9) and (A12) and integrating over x_1, x_2, we obtain from (A4) and (A11) that

$$\sum_{\sigma,\sigma'} \overline{\langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda'''\sigma'''\rangle}\langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda''\sigma''\rangle = \delta_{\Lambda''\Lambda'''}\, \delta_{\sigma''\sigma'''}. \tag{A13}$$

The Clebsch–Gordan coefficients could be written in matrix element form with the help of the definitions

$$\overline{\langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda'''\sigma'''\rangle} \equiv \langle\Lambda'''\sigma''' \mid \bar{M}^{\Lambda\Lambda'} \mid \sigma\sigma'\rangle, \tag{A14a}$$

$$\langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda''\sigma''\rangle \equiv \langle\sigma\sigma' \mid M^{\Lambda\Lambda'} \mid \Lambda''\sigma''\rangle, \tag{A14b}$$

so that (A13) in matrix notation takes the form

$$\bar{\mathbf{M}}^{\Lambda\Lambda'}\mathbf{M}^{\Lambda\Lambda'} = \mathbf{I}, \tag{A15}$$

where I is the unit matrix of elements $\delta_{\Lambda''\Lambda'''}\delta_{\sigma''\sigma'''}$. We note that the number of values that σ, σ' can take, which is $d(\Lambda)d(\Lambda')$ where $d(\Lambda)$ is the dimension of the representation, is the same as that of the values $\Lambda''\sigma''$, taking into account all resulting representations from the combination of Λ and Λ' and all values of their rows. Thus the matrices in (A14) are square ones, and, as a left inverse is then also a right inverse, we get

$$\mathbf{M}^{\Lambda\Lambda'}\bar{\mathbf{M}}^{\Lambda\Lambda'} = \mathbf{I} \tag{A16}$$

and thus

$$\sum_{\Lambda''\sigma''} \langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda''\sigma''\rangle\overline{\langle\Lambda\bar{\sigma}, \Lambda'\bar{\sigma}' \mid \Lambda''\sigma''\rangle} = \delta_{\sigma\bar{\sigma}}\, \delta_{\sigma'\bar{\sigma}'}. \tag{A17}$$

From (A17) we obtain immediately that

$$\chi_\sigma^\Lambda(x_1)\, \chi_{\sigma'}^{\Lambda'}(x_2) = \sum_{\Lambda''\sigma''} \overline{\langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda''\sigma''\rangle}\, X_{\sigma''}^{\Lambda\Lambda'\Lambda''}(x_1, x_2). \tag{A18}$$

We now proceed to derive the Wigner–Eckart theorem when we deal with nonorthonormal bases. Let us consider an irreducible tensor T_σ^Λ defined by the transformation properties

$$P_R T_\sigma^\Lambda P_{R^{-1}} = \sum_{\bar{\sigma}} T_{\bar{\sigma}}^\Lambda \Delta_{\bar{\sigma}\sigma}^\Lambda(R), \tag{A19}$$

and consider the evaluation of the integral

$$\int \bar{\chi}_{\sigma''}^{\Lambda''*}(T_\sigma^\Lambda \chi_{\sigma'}^{\Lambda'})\, dx \equiv \langle\Lambda''\sigma'' \mid T_\sigma^\Lambda \mid \Lambda'\sigma'\rangle. \tag{A20}$$

We first look at the auxiliary expression

$$\int \bar{\chi}_{\sigma''}^{\Lambda''*}[T^\Lambda \times \chi^{\Lambda'}]_{\sigma'''}^{\Lambda'''}\, dx, \tag{A21a}$$

where

$$[T^\Lambda \times \chi^{\Lambda'}]_{\sigma'''}^{\Lambda'''} = \sum_{\sigma,\sigma'} \langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda'''\sigma'''\rangle\, T_\sigma^\Lambda \chi_{\sigma'}^{\Lambda'}. \tag{A21b}$$

From (A19), (A7), (A8) we then have that integral (A21) is equal to

$$\begin{aligned} &\int (P_R\bar{\chi}_{\sigma''}^{\Lambda''})^* \{P_R[T^\Lambda \times \chi^{\Lambda'}]_{\sigma'''}^{\Lambda'''}\}\, dx \\ &= \sum_{\bar{\sigma}''\bar{\sigma}'''} \Delta_{\sigma''\bar{\sigma}''}^{\Lambda''}(R^{-1})\, \Delta_{\bar{\sigma}'''\sigma'''}^{\Lambda'''}(R) \int \bar{\chi}_{\bar{\sigma}''}^{\Lambda''*}[T^\Lambda \times \chi^{\Lambda'}]_{\bar{\sigma}'''}^{\Lambda'''}\, dx. \end{aligned} \tag{A22}$$

Multiplying then both sides by the volume element dR of the group parameters and integrating over it, we obtain

$$\begin{aligned} &\int \bar{\chi}_{\sigma''}^{\Lambda''*}[T^\Lambda \times \chi^{\Lambda'}]_{\sigma'''}^{\Lambda'''}\, dx \\ &= \delta_{\Lambda''\Lambda'''}\delta_{\sigma''\sigma'''}[d(\Lambda'')]^{-1} \int \sum_\sigma \bar{\chi}_\sigma^{\Lambda''*}[T^\Lambda \times \chi^{\Lambda'}]_\sigma^{\Lambda''}\, dx, \end{aligned} \tag{A23}$$

where we made use of the relation[8]

$$\begin{aligned} &\int \Delta_{\sigma''\bar{\sigma}''}^{\Lambda''}(R^{-1})\, \Delta_{\bar{\sigma}'''\sigma'''}^{\Lambda'''}(R)\, dR \\ &= [d(\Lambda'')]^{-1}\, \delta_{\Lambda''\Lambda'''}\delta_{\sigma''\sigma'''}\delta_{\bar{\sigma}''\bar{\sigma}'''} \int dR, \end{aligned} \tag{A24}$$

which is a consequence of Schur's lemma and independent of whether the representation is unitary or not.

Obviously the last integral in (A23) is a function of Λ, Λ', Λ'' only which we could designate by

$$[d(\Lambda'')]^{-1} \int \sum_\sigma \bar{\chi}_\sigma^{\Lambda''*}[T^\Lambda \times \chi^{\Lambda'}]_\sigma^{\Lambda''}\, dx \equiv \langle\Lambda'' \| T^\Lambda \| \Lambda'\rangle, \tag{A25}$$

so that, using the orthonormality property (A17) of the Clebsch–Gordan coefficients and their duals, we obtain

$$\begin{aligned} &\int \bar{\chi}_{\sigma''}^{\Lambda''*}(T_\sigma^\Lambda \chi_{\sigma'}^{\Lambda'})\, dx \\ &= \overline{\langle\Lambda\sigma, \Lambda'\sigma' \mid \Lambda''\sigma''\rangle}\langle\Lambda'' \| T^\Lambda \| \Lambda'\rangle. \end{aligned} \tag{A26}$$

This last result is the derivation of the Wigner–Eckart theorem by essentially the standard method in which we were only careful to note that our basis is not orthonormal and thus we have to use also the duals of all the concepts involved such as bases, representations, and Clebsch–Gordan coefficients.

We now particularize all the above results to symmetric representations of the O(5) group, i.e., $\Lambda = (\lambda 0)$ so we can replace Λ by λ, σ by the ρ of (A1) and x is given by the angles in (A2). We have then

$$\int \bar{\chi}_{\rho''}^{\lambda''*}(T_\rho^\lambda \chi_{\rho'}^{\lambda'})\, dx = \overline{\langle\lambda\rho, \lambda'\rho' \mid \lambda''\rho''\rangle}\langle\lambda'' \| T^\lambda \| \lambda'\rangle, \tag{A27}$$

but we still need to determine the Clebsch–Gordan coefficient in (A27). We note, though, that from (A18)

$$\chi_\rho^\lambda(x_1)\, \chi_{\rho'}^{\lambda'}(x_2) = \sum_{\Lambda''\sigma''} \overline{\langle\lambda\rho, \lambda'\rho' \mid \Lambda''\sigma''\rangle}\, X_{\sigma''}^{\lambda\lambda'\Lambda''}(x_1, x_2) \tag{A28}$$

and that this relation holds no matter what the x_1, x_2. In particular, it is valid if $x_1 = x_2 = x$, in which case $X_{\sigma''}^{\lambda\lambda'\Lambda''}(x, x)$ vanishes unless[7] $\Lambda'' = (\lambda''0)$, i.e., we have only the symmetric representations. From the transformation properties (A10) of this last function we conclude that

$$\mathrm{X}_{\sigma''}^{\lambda\lambda'\Lambda''}(x,x) = \begin{cases} G(\lambda\lambda'\lambda'')\chi_{\rho''}^{\lambda''}(x) & \text{if } \Lambda''=(\lambda'',0),\ \sigma''=\rho'', \\ 0 & \text{otherwise,} \end{cases} \tag{A29}$$

where $G(\lambda\lambda'\lambda'')$ is so far an undetermined function of the indicated variables. Substituting this result in (A28), we obtain from the dual property (A4) that

$$\overline{\langle\lambda\rho,\lambda'\rho'\,|\,\lambda''\rho''\rangle} = G^{-1}(\lambda\lambda'\lambda'')\{\lambda\rho,\lambda'\rho'\,|\,\lambda''\rho''\}, \tag{A30}$$

where the last coefficient is given by (3.11) if we take into account the definition of ρ in (A1).

Thus (A27), (A30) give the Wigner—Eckart theorem in precisely the form we need it in Secs. 4 and 5.

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The groups of Poincaré and Galilei in arbitrary dimensional spaces

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In arbitrary dimensional spaces the Lie algebra of the Poincaré group is seen to be a subalgebra of the complex Galilei algebra, while the Galilei algebra is a subalgebra of Poincaré algebra. The usual contraction of the Poincaré to the Galilei group is seen to be equivalent to a certain coordinate transformation.

I. INTRODUCTION

It is well known that the Galilei group in $2+1$ dimensions is a subgroup of that of Poincaré. A beautiful way to see this is by means of a change of coordinates[1] which is usually called the light-cone transformation. This fact is physically understood as the result of an infinite boost of a system in some direction of the space, which leads to the loss of the spatial dimension in this direction—due to the Lorentz contraction—and leaves the remaining system with a Galilean structure. This method has had several applications, for instance, to the study of the internal structure of hadrons at very high energies[2] and to the connection of relativistic and Galilean field equations for arbitrary spin particle.[3] Moreover, with a modification of the light-cone transformation involving a continuous parameter[4] it has been possible to go from the $(2+1)$-dimensional Galilei group to the Poincaré one in the same dimension, in a procedure inverse to the ordinary contraction of the Poincaré to the Galilei group when $c\to\infty$.

On the other hand, it has recently been shown that the ordinary Lie algebra of the Poincaré group is a subalgebra[5] of the complex Lie algebra of the Galilei group in $4+1$ dimensions. Also this result has been obtained through an adequate change of coordinates. Nevertheless its physical interpretation is not so clear. One possible application of this connection is the derivation of relativistic equations starting with Galilean ones, in just the reciprocal way of the former case.[6]

Summing up, the complex Galilei algebra in $4+1$ dimensions contains the ordinary Poincaré algebra, which in turn contains a $(2+1)$-dimensional Galilei algebra, and these relations have their parallel counterpart at the level of the corresponding wave equations. The generalization of this situation to an arbitrary number n of space dimensions is one of the objects of the present paper. In this way we shall see that for an abstract physical theory the election of one or another invariance group (i.e., Poincaré or Galilei) is not so fundamental as one would think, because it is possible to obtain a relativistic theory in $(n-1)+1$ dimensions from a Galilean one in $n+1$ dimensions, and vice-versa.

Another purpose of this work is to investigate which is in the present case the modification of the coordinate transformation[5] which enables us to obtain the ordinary Galilei group from the Poincaré one in the same dimension, as well as the relation of this modificated coordinate transformation with the usual contraction of the Poincaré to the Galilei group. In other words, we want to study the corresponding procedure to that of the quasi-light-cone frame[4] in the present case.

The organization of the paper is as follows: In Sec. 2, we generalize the coordinate transformation of Ref. 5 and see how the Poincaré algebra in n space/1 time dimensions is a subalgebra of the complex Galilei algebra in $(n+1)$ space/1 time coordinates. In Sec. 3 we generalize the light-cone frame transformation to an arbitrary number of dimensions of the space. In Sec. 4 we develop a parametrization of the original coordinate transformation which enables us to pass from the Poincaré to the Galilei group in the same number of dimensions. We study its relation with the ordinary contraction $c\to\infty$. Section 5 is devoted to conclusions.

2. P_{n+1} AS A SUBGROUP OF $G_{(n+1)+1}$

The Lie algebra of the extended Galilei group in $(n+1)$ space/1 time dimensions is given by

$$\begin{aligned}
&[L_{rs},L_{uv}]=i(\delta_{rv}L_{su}+\delta_{su}L_{rv}-\delta_{ru}L_{sv}-\delta_{sv}L_{ru}),\\
&[L_{rs},G_u]=-i(\delta_{ru}G_s-\delta_{su}G_r),\\
&[L_{rs},P_u]=-i(\delta_{ru}P_s-\delta_{su}P_r),\\
&[G_r,G_s]=[P_r,P_s]=[L_{rs},H]=[P_r,H]=0,\\
&[G_r,H]=iP_r\ ,\\
&[G_r,P_s]=i\delta_{rs}\mu\\
&\quad(r,s,u,v=1,2,\cdots,n+1)\ ,\\
&[L_{rs},\mu]=[P_r,\mu]=[G_r,\mu]=[H,\mu]=0\ ,
\end{aligned}\tag{2.1}$$

where L_{rs} are the generators of rotations, G_r the generators of Galilean boosts, P_r those of the space translations, H the generator of time translations, and μ is the neutral element of the algebra, which is associated with the mass.

The generalization of the coordinate transformation introduced in Ref. 5 to this case is the following. If $x^\alpha=(x^0,x^1,\cdots,x^{n+1})$ are the old and $\bar{x}^\alpha=(\bar{x}^0,\bar{x}^1,\cdots,\bar{x}^{n+1})$ the new coordinates of an arbitrary point in the space–time—x^0 being the time coordinate—they are related by

$$\begin{aligned}
&\bar{x}^0=ix^{n+1},\\
&\bar{x}^i=x^i\quad(i=1,2\cdots,n),\\
&\bar{x}^{n+1}=x^0\gamma(\gamma\ \textit{arbitrary}).
\end{aligned}\tag{2.2}$$

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With this transformation the real Lie algebra (2.1) becomes a complex algebra of Galilei group. Let us consider the following subset of the transformed generators:

$$
\begin{aligned}
&m_{ij}\equiv\overline{-L}_{ij}=-L_{ij},\\
&K_i\equiv\overline{L^{0i}}=-iL_{n+1,i},\\
&d_i\equiv\bar{P}_i=P_i \quad (i,j+1,2,\ldots,\mu),\\
&h\equiv P_0=-iP_{n+1},
\end{aligned}
\tag{2.3}
$$

The reason for the selection of these generators has already been discussed in Ref. 6. The commutation relations for these generators are the following:

$$
\begin{aligned}
&[m_{ij},m_{kl}]=-i(\delta_{il}m_{jk}+\delta_{jk}m_{il}-\delta_{ik}m_{jl}-\delta_{jl}m_{ik}),\\
&[m_{ij},k_l]=i(\delta_{il}k_j-\delta_{jl}k_i),\\
&[m_{ij},d_l]=i(\delta_{il}d_j-\delta_{jl}d_i),\\
&[k_i,k_j]=-im_{ij},\\
&[d_i,d_j]=[m_{ij},h]=[d_i,h]=0,\\
&[k_i,h]=id_i, \quad [k_i,d_j]=i\delta_{ij}h,
\end{aligned}
\tag{2.4}
$$

which constitute the Lie algebra of the Poincaré group in n space/1 time dimensions.

3. LIGHT-CONE TRANSFORMATION IN ANY NUMBER OF DIMENSIONS

The commutation relations of the $(n+1)$-dimensional Poincaré group which we have just obtained can be written in the compact form

$$
\begin{aligned}
&[M_{\mu\nu},M_{\rho\sigma}]=i(g_{\mu\sigma}M_{\nu\rho}+g_{\nu\rho}M_{\mu\sigma}\\
&\qquad -g_{\mu\rho}M_{\nu\sigma}-g_{\nu\sigma}M_{\mu\rho}),\\
&[P_\mu,M_{\rho\sigma}]=i(g_{\mu\rho}P_\sigma-g_{\mu\sigma}P_\rho),\\
&[P_\mu,P_\nu]=0 \quad (\mu,\nu,\rho,\sigma=0,1,\ldots,n),
\end{aligned}
\tag{3.1}
$$

where $K_i=M_{i0}$ $(i=1,2,\cdots,n)$ are the generators of boosts, M_{ij} $(i,j=1,2,\cdots,n)$ those of the rotations, P_i those of the space translations, and $H=\bar{P}_0$ the generator of the time translations. The metric tensor is

$$
\bar{g}=\begin{bmatrix} +1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}_{n}.
$$

The generalization of the ordinary light-cone frame transformation to the present case is

$$
\begin{aligned}
&\bar{\bar{x}}^0=(\bar{x}_0+\bar{x}_n)/\sqrt{2},\\
&\bar{\bar{x}}^a=\bar{x}^a \quad (a=1,2,\cdots,n-1)\\
&\bar{\bar{x}}^n=(\bar{x}_0-\bar{x}_n)/\sqrt{2},\\
&\bar{\bar{x}}^{n+1}=\bar{x}^{n+1}.
\end{aligned}
\tag{3.2}
$$

The metric tensor is transformed into the following:

$$
\bar{\bar{g}}=\begin{bmatrix} 0 & & & & 1 \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ 1 & & & & 0 \end{bmatrix},
$$

while the new $M_{\mu\nu}$ and P_μ are given by

$$
\bar{\bar{P}}_\mu=((\bar{P}_0+\bar{P}_n)/\sqrt{2},P_1,\ldots,P_{n-1},(\bar{P}_0-\bar{P}_n)/\sqrt{2}),
$$

$$
\bar{\bar{M}}_{\mu\nu}=\begin{pmatrix}
0 & -(K_1+M_{1n})/\sqrt{2} & -(K_2+M_{2n})/\sqrt{2} & \cdots & K_n\\
(K_1+M_{1n})/\sqrt{2} & 0 & M_{12} & \cdots & (K_1-M_{1n})/\sqrt{2}\\
(K_2+M_{2n})/\sqrt{2} & -M_{12} & 0 & \cdots & (K_2-M_{2n})/\sqrt{2}\\
\vdots & \vdots & \vdots & & \vdots\\
-K_n & (-K_1+M_{1n})/\sqrt{2} & (-K_2+M_{2n})/\sqrt{2} & \cdots & 0
\end{pmatrix}.
\tag{3.3}
$$

Let us consider now the following subset of the new generators:

$$
\begin{aligned}
&l_{ab}\equiv-M_{ab}, && h\equiv(\bar{P}_0+\bar{P}_n)/\sqrt{2},\\
&g_a\equiv(K_a-M_{an})/\sqrt{2}, && \eta=(\bar{P}_0-\bar{P}_n)/\sqrt{2},\\
&d_a\equiv P_a, && (a.b=1,2,\cdots,n).
\end{aligned}
\tag{3.4}
$$

They satisfy the following commutation relations:

$$
\begin{aligned}
&[l_{ab},l_{cd}]=i(\delta_{ad}l_{bc}+\delta_{bc}l_{ad}-\delta_{ac}l_{bd}-\delta_{bd}l_{ac}),\\
&[l_{ab},g_c]=-i(\delta_{ac}g_b-\delta_{bc}g_a),\\
&[l_{ab},d_c]=-i(\delta_{ac}d_b-\delta_{bc}d_a),\\
&[g_a,g_b]=[d_a,d_b]=[l_{ab},h]=[d_a,h]=0,\\
&[g_a,h]=id_a, \quad [g_a,d_b]=i\delta_{ab}\eta,\\
&[l_{ab},\eta]=[d_a,\eta]=[g_a,\eta]=[h,\eta]=0\\
&\qquad (a,b,c,d=1,2,\ldots,n-1),
\end{aligned}
\tag{3.5}
$$

which constitute the Lie algebra of the Galilei group in $(n-1)$ space/1 time dimensions.

4. THE CONTRACTION $c\to\infty$ AS A COORDINATE TRANSFORMATION

In this section we consider the case $n=4$ for simplicity, but all the results are immediately generalizable to arbitrary n. Let us start with the commutation relations (2.1) in the particular case $n=4$. Consider now the following coordinate transformation:

$$
\begin{aligned}
&\bar{x}^0=\alpha x^0+\beta x^4,\\
&\bar{x}^i=x^i \quad (i=1,2,3),\\
&\bar{x}^4=\gamma x^0+\delta x^4.
\end{aligned}
\tag{4.1}
$$

The commutation relations for the generators

$$
\bar{l}_i=l_i\equiv-\tfrac{1}{2}\epsilon_{ijk}L_{jk}, \quad \bar{d}_i=d_i
$$

$$\bar{d}_0=\frac{\delta}{\alpha\delta-\beta\gamma}d_0-\frac{\gamma}{\alpha\delta-\beta\gamma}d_4,$$
$$k_i\equiv\overline{M^{0i}}=-(\alpha g_i+\beta\lambda_i)$$
$$(i,j,k=1,2,3)\quad(\lambda_i\equiv L_{4i}),$$

are given by

$$\begin{aligned}&[l_i,l_j]=i\epsilon_{ijk}l_k,\quad [l_i,k_j]=i\epsilon_{ijk}k_k,\\&[l_i,d_j]=i\epsilon_{ijk}d_k,\quad [k_i,k_j]=ib^2\epsilon_{ijk}l_k,\\&[d_i,d_j]=[l_i,h]=[d_i,h]=0,\\&[k_i,h]=-i\Big(\frac{\alpha\delta}{\alpha\delta-\beta\gamma}+\frac{\beta\gamma}{\alpha\delta-\beta\gamma}\Big)d_i,\\&[k_i,d_j]=-i(\alpha\mu+\beta^2h+\beta\delta d_4)\delta_{ij}.\end{aligned}\tag{4.2}$$

Depending on the values of the constants α, β, γ, δ, these relations either constitute the algebra of the Poincaré group or the algebra of the Galilei group. In fact, for

$$\alpha=0,\quad \beta=\pm i,\quad \gamma \text{ arbitrary},\quad \delta=0.\tag{4.3}$$

The relations (4.2) turn out to be the (3.1) in the case $n=3$, while for

$$\alpha=1,\quad \beta=0,\quad \gamma \text{ and } \delta \text{ arbitrary}.\tag{4.4}$$

Equations (4.2) are the same as (2.1) in the case $n=2$ (with $g_i\equiv-k_i$).

The coordinate transformation can be parametrized in order to include these two particular cases for different values of the parameter. In fact, the following transformation:

$$\bar{x}=T(\epsilon,\gamma)x,\quad T(\epsilon,\gamma)\equiv\begin{pmatrix}\cosh i\epsilon & 0 & \sinh i\epsilon\\ 0 & I_3 & 0\\ \gamma & 0 & 0\end{pmatrix}.\tag{4.5}$$

reduces to (4.3) when $\epsilon=\pm\pi/2$ and to (4.4) when $\epsilon=0$.

On the other hand, in order to compare this procedure with the usual contraction of the Poincaré to the Galilei group when $c\to\infty$, we now confront (4.2) with the commutation relations obtained from the Poincaré algebra after the application of the limiting process, but where the terms up to first order in $1/c^2$ are still taken into account[7]:

$$\begin{aligned}&[L_i,L_j]=i\epsilon_{ijk}L_k,\quad [L_i,K_j']=i\epsilon_{ijk}K_k',\\&[L_i,P_j]=i\epsilon_{ijk}P_k,\quad [P_i,P_j]=[L_i,H']=[P_i,H']=0,\\&[K_i',H']=iP_i,\quad [K_i',K_j']=-(i/c^2)\epsilon_{ijk}L_k,\\&[K_i',P_j]=(i/c^2)\delta_{ij}H'+i\mu\delta_{ij}.\end{aligned}$$

We see that (4.6) can be obtained from (4.2) provided we put

$$\alpha=1,\quad \beta=\pm i/c,\quad \gamma \text{ arbitrary},\quad \delta=0.$$

This transformation is also obtained from (4.5) when $\epsilon=\pm1/c$. Notice that the arbitrarity of δ in (4.4) is only attained when $\epsilon=0$. Therefore it is not restrictive to put $\delta=0$, in general, as we have done in (4.5) to define the parametrization.

Within the parametrization (4.5) we have been able to reproduce the usual contraction $c\to\infty$, both when first-order terms in $1/c^2$ are considered and also when the limit is fully applied. Observe that the full contraction corresponds to a finite jump from $\pm\pi/2$ to 0 of the parameter ϵ, while the supression of the terms of first order in $1/c^2$ (i.e., the last step of the full contraction) only amounts to a correspondingly infinitesimal change from $\epsilon=\pm1/c$ to $\epsilon=0$.

5. CONCLUSIONS

We have seen that Poincaré algebra is a subalgebra of complex Galilei algebra while the Galilei algebra is a subalgebra of the ordinary Poincaré algebra, if we consider them in arbitrary dimensional space. And this has been shown by means of very simple linear changes of coordinates: the light-cone transformation[1] in one case and an imaginary coordinate transformation[5] in the other. The physical relevance of this result lies in the fact that, whatever be the invariance group of the physical theory that we take at the beginning (Poincaré or Galilei) it is always possible to extract a theory, in a space with one dimension less, invariant under the other group.

Making use of a convenient parametrization of the imaginary transformation,[5] we realized that the usual procedure of contraction of the Poincaré to the Galilei group can be put into the form of a change of the space–time coordinates.

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Direct-inverse problems in transport theory. 1. The inverse problem[a)]

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For the inverse problem treated here we use the results of an experiment which measures the total angular-dependent column density as compared to the measurement which provides information on the angle integrated spatial-dependent angular density (or the specific intensity). We use two methods of approach. One, the Legendre expansion method and two a maximal variational principle. In particular we demonstrate how the variational principle yields a very convenient representation of the scattering kernel (or the phase function) in terms of a basis consisting of Case eigenfunctions for the isotropically scattering medium.

1. INTRODUCTION

The object of the direct problem of transport theory, (e.g., neutron or radiative) the usual problem treated, is to find the distribution function for a given scattering function. By contrast, the inverse problem involves construction of the scattering function from the results of some simple experiment which gives some knowledge of the distribution function. For the inverse problem in general, it is usually not clear with regard to the nature of the minimal set of measurements one needs in order to construct the scattering function uniquely. For instance, consider a relatively simple case of transport of monoenergetic neutrons in a medium with a homogeneous mixture of different nuclear species. If the angular scattering differential cross section associated with each species is different, then it would appear to be difficult to devise an experiment from which any set of measurements can lead to a unique conclusion for the shape of the angular differential cross section for each nuclear species. If one had a medium consisting of only one nuclear species, which is not necessarily homogeneously distributed, then certain relatively simple measurements will lead to a unique determination of the differential cross section. The reason is simply that in that case the secondary production function, which is the ratio of the rate at which the neutrons are elastically scattered to the rate at which the scattering involves all nuclear processes, become independent of the density of that nuclear species and depends only on its scattering properties. Similar considerations apply to problems of radiative transfer and, *inter alia*, the problem of transport of hyperthermal electrons in a plasma containing a substantial amount of neutral species of various kinds.

2. DIRECT INVERSE PROBLEMS

For our presentation of the direct inverse problems in transport theory we consider an infinite medium with a plane source at $x=x_0$ emitting one-speed neutrons in a direction whose cosine is μ_0. For the one-dimensional time-independent, and azimuthally symmetric case, the standard neutron transport equation is[1]

$$\mu \frac{\partial \Psi}{\partial x}(x,\mu\to\mu_0)+\Psi=\frac{c}{2}\int_{-1}^{1} d\mu' f(\mu'\to\mu)\Psi(x,\mu'\to\mu_0) + \delta(x-x_0)\delta(\mu-\mu_0) \qquad (1)$$

where, $c<1$ (nonmultiplying medium), Ψ is the distribution function and $f(\mu'\to\mu)$ is the normalized symmetric scattering function, i.e.,

$$\tfrac{1}{2}\int_{-1}^{1} d\mu' f(\mu'\to\mu)=1. \qquad (2)$$

For the direct problem, $f(\mu'\to\mu)$ is given and Ψ is to be determined everywhere. For the inverse problem certain results of an experiment are given and then $f(\mu'\to\mu)$ is to be constructed from those measurements.

In an earlier paper,[2] Case presents an elegant method of solving the inverse problem. The essence of his procedure is that, if one expands the scattering function in terms of Legendre polynomials so that

$$f(\mu'\to\mu)=\sum_{l=0}^{\infty}(2l+1)f_l P_l(\mu')P_l(\mu), \qquad (3)$$

then for the unit isotropic plane source the expansion coefficients

$$g(l)=1-cf_l \qquad (4)$$

are determined from the measurement of the density of neutrons as a function of x. Therefore, as Case has shown from the direct problem, one has the spectral representation of the total density $[\Phi(x)]$

$$\Phi(x)=\int_0^{\infty}\frac{d\rho(\nu)}{\nu}e^{-|x|/\nu}, \qquad (5)$$

where $\rho(\nu)$ is the spectral function given by

$$\frac{d\rho(\nu)}{\nu}=\frac{d\nu}{N(\nu)}, \quad -1\leqslant\nu\leqslant 1$$
$$=\sum_i \frac{\delta(\nu-\nu_i)d\nu}{N_i}, \quad |\nu|>1, \qquad (6)$$

where $N(\nu)$ and N_i are the normalizations of the eigenfunctions of the continuous and discrete spectra, respectively. For the inverse problem he states that, given the measurement of the total density $\Phi(x)$, one knows quite a bit about the spectral density from Eq. (5). We will not give further

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details here as to how this argument actually leads to the determination of the expansion coefficients $g(l)$ and hence $f(\mu'\to\mu)$ (see Refs. 2 and 3 for a complete discussion).

We wish to present a different method in the present paper for dealing with the inverse problem. In contrast to the data used in Case's approach we require the measurement of the angular column density $M_0(\mu,\mu_0)$ of neutrons for a unit plane source emitting neutrons in one direction μ_0 (monodirectional). In other words, the experiment should provide the information on the quantity

$$M_0(\mu,\mu_0)=\int_{-\infty}^{\infty} dx\, \Psi(x,\mu\to\mu_0) \tag{7}$$

where $\Psi(x,\mu\to\mu_0)$ is the angular density of neutrons satisfying Eq. (1).

Clearly $M_0(\mu,\mu_0)$ is the angle-dependent zeroth moment of the distribution function (the angular density) which is related to the scattering function. That relation is readily obtained from Eq. (1) by integration with respect to x. With the appropriate boundary conditions that Ψ vanishes at $x=\pm\infty$, we have from Eq. (1)

$$M_0(\mu,\mu_0)=\delta(\mu-\mu_0)+\frac{c}{2}\int_{-1}^{1} d\mu'\, f(\mu'\to\mu)M_0(\mu',\mu_0). \tag{8}$$

This is a Fredholm type of an integral equation for $M_0(\mu,\mu_0)$ which involves the scattering function as the kernel. If we let

$$S_0(\mu,\mu_0)=\int_{-1}^{1} d\mu'\, f(\mu'\to\mu)M_0(\mu',\mu_0) \tag{9}$$

then $S_0(\mu,\mu_0)$ (the emission term) also satisfies a similar type of an integral equation. That equation is readily obtained by multiplying Eq. (8) by $f(\mu\to\mu_1)$ and integrating with respect to μ. With change of names of variables we have

$$S_0(\mu,\mu_0)=f(\mu_0\to\mu)+\frac{c}{2}\int_{-1}^{1} d\mu'\, f(\mu'\to\mu)S_0(\mu',\mu_0). \tag{10}$$

For convenience we write Eq. (8) as

$$M_0(\mu,\mu_0)=\delta(\mu-\mu_0)+\frac{c}{2}S_0(\mu,\mu_0)\,. \tag{11}$$

Before addressing the Direct Inverse problems, we wish to point out that if in the integral equations (8) and (10) the kernel $f(\mu\to\mu_0)$ satisfies certain properties then there is a maximal variational principle[4] which can be used to solve such equations. Explicitly, if $f(\mu\to\mu_0)$ satisfies the following properties:

(a) $f(\mu\to\mu_0)$ is symmetric in μ,μ_0,

(b) $f(\mu\to\mu_0)\geqslant 0$, for $-1\leqslant(\mu,\mu_0)\leqslant 1$,

(c) $\frac{c}{2}\int_{-1}^{1} d\mu'\, f(\mu\to\mu_0)\leqslant 1,\ -1\leqslant\mu_0\leqslant 1,\ c<1$,

then the functional

$$F_S[n]=\int_{-1}^{1} d\mu\, n(\mu,\mu_0)\Big[\,2f(\mu,\mu_0)-n(\mu,\mu_0) +\frac{c}{2}\int_{-1}^{1} d\mu'\, f(\mu'\to\mu)n(\mu',\mu_0)\Big] \tag{12}$$

is an *absolute maximum* if and only if $n(\mu,\mu_0)$ is an exact solution of the integral equation (10). Or conversely, as in the inverse problem, if we consider Eq. (10) as an integral equation for $f(\mu_0\to\mu)$ so that

$$f(\mu_0\to\mu)=S_0(\mu,\mu_0)-\frac{c}{2}\int_{-1}^{1} d\mu'\, S_0(\mu',\mu_0)f(\mu'\to\mu), \tag{13a}$$

then the functional

$$F_f[n]=\int_{-1}^{1} d\mu\, n(\mu_0,\mu)\Big[\,2S_0(\mu,\mu_0)-n(\mu_0,\mu) -\frac{c}{2}\int_{-1}^{1} d\mu'\, S_0(\mu',\mu_0)n(\mu',\mu)\Big] \tag{13b}$$

is an absolute maximum if and only if $n(\mu_0,\mu)=f(\mu_0\to\mu)$ is the exact solution of the integral equation (13a). This result follows from the fact that $S_0(\mu,\mu_0)$ is symmetric and the operator Λ corresponding to the integral equation (13a), i.e.

$$\Lambda f=f \tag{13c}$$

is positive definite. In two previous papers by Kanal and Moses[5,6] such a variational principle was used to solve the problems of inverse scattering and also a demonstration of its application to the linear transport theory for the isotropic scattering inhomogeneous media was given. Here for the direct inverse problems we can proceed in two ways. One is to expand $f(\mu\to\mu_0)$ in terms of Legendre polynomials and use their orthogonality property to relate the expansion coefficients to the Legendre moments of $S_0(\mu,\mu_0)$. This method produces exact solutions. The other way is to use the variational principle. Use of either method will be dictated by the kind of an experiment that is required to relate the scattering function $f(\mu\to\mu_0)$ to the emission term $S_0(\mu,\mu_0)$ or vice versa. We shall illustrate both methods.

A. Inverse Problem by Legendre Expansion

First we note that if $f(\mu\to\mu_0)$ is symmetric then so is $S_0(\mu,\mu_0)$. Let us now expand $f(\mu\to\mu_0)$ in terms of Legendre polynomials so that

$$f(\mu\to\mu_0)=\sum_{n=0}^{\infty}(2n+1)f_nP_n(\mu)P_n(\mu_0). \tag{14}$$

Insert this expansion in Eq. (10) to obtain

$$S_0(\mu,\mu_0)=\sum_{n=0}^{\infty}(2n+1)f_nP_n(\mu)P_n(\mu_0). +\frac{c}{2}\sum_{n=0}^{\infty}(2n+1)f_nP_n(\mu)q_n(\mu_0), \tag{15}$$

where

$$q_n(\mu_0)=\int_{-1}^{1} d\mu\, P_n(\mu)S_0(\mu,\mu_0). \tag{16}$$

Using the orthogonality property of Legendre polynomials

$$\int_{-1}^{1} d\mu\, P_n(\mu)P_l(\mu)=\frac{2}{2n+1}\,\delta nl, \tag{17}$$

we conclude from Eq. (15) that

$$f_l=\frac{q_l(\mu_0)}{2P_l(\mu_0)+cq_l(\mu_0)}. \tag{18}$$

Note that f_l are independent of μ_0. In consequence one needs to make the measurement for any one direction μ_0 to determine all the expansion coefficients f_l. In particular, for $\mu_0=1$, i.e., for the source emitting neutrons parallel to the x axis, we have

$$f_l=\frac{q_l(1)}{2+cq_l(1)}. \tag{19}$$

This result is really not surprising for a homogeneous medium due to the fact that the medium is rotationally invariant. However, it is of prime importance for the construction of $f(\mu\to\mu_0)$. In particular, if the medium is not drastically anisotropically scattering, then one needs only a few expansion coefficients and this method becomes very useful.

Now the Legendre moments $q_l(\mu_0)$ of $S_0(\mu,\mu_0)$ can be determined from the measurement of the angular column density $M_0(\mu,\mu_0)$. In other words, given $M_0(\mu,\mu_0)$, we conclude from Eq. (11), that

$$q_l(\mu_0)=\frac{2}{c}\left[\int_{-1}^{1} d\mu\, P_l(\mu)M_0(\mu,\mu_0)-P_l(\mu_0)\right]. \tag{20}$$

The quantity c (the secondary production term) is obtained from Eq. (8) [or from Eq. (11)] by integration with respect to μ so that

$$c=1-\left[\int_{-1}^{1} d\mu\, M_0(\mu,\mu_0)\right]^{-1}. \tag{21}$$

This is a well-known result (cf. Ref. 1). However, from the inverse problem point of view we again note that c, being independent of μ_0, is determined from the experiment with the source emitting neutrons in any arbitrary direction.

It would be interesting to know the degree of anisotropy of the scattering medium. We can estimate that from the experiment if $q_l(\mu_0)$ were calculated for all μ_0, i.e., if we calculate all the Legendre moments of the column density [see for example Eq. (19)]. For in that case we obtain from Eq. (18) the relation

$$\int_{-1}^{1} d\mu\, P_n q_l(\mu)=\frac{4f_n}{(2n+1)(1-cf_n)}\,\delta nl. \tag{22}$$

In other words, $(P_n(\mu), q_l(\mu))$ form a biorthogonal set, but more importantly for $l=n$ we get

$$f_n=\frac{(2n+1)T_n}{4+c(2n+1)T_n}, \tag{23}$$

where

$$T_n=\int_{-1}^{1} d\mu\, P_n(\mu)q_n(\mu). \tag{24}$$

The f_n's will give a measure of the degree of anisotropy so that $f_n=0$, for $n\geqslant N$.

B. Inverse Problem by the Variational Principle

Expansion in terms of Legendre polynomials is useful when the situation is not too anisotropic. When one wishes to consider very anisotropic cases, the variational principle may be more useful. Thus, consider Eq. (13a) so that

$$f(\mu\to\mu_0)=S_0(\mu,\mu_0)-\frac{c}{2}\int_{-1}^{1} d\mu'\, S_0(\mu',\mu_0)f(\mu'\to\mu). \tag{25}$$

If $n(\mu,\mu_0)=f(\mu\to\mu_0)$ is the exact solution of Eq. (25), then the functional defined by (13b) is an absolute maximum. The value of the functional is

$$F_f[f]=\frac{2}{c}\,[S_0(\mu_0,\mu_0)-f(\mu_0\to\mu_0)] \tag{26a}$$

or

$$f_N(\mu_0\to\mu_0)\simeq S_0(\mu_0,\mu_0)-\frac{c}{2}F_{f_N}[f_N], \tag{26b}$$

where f_N is a class of trial functions. From (26a) we see that from the class of functions $\{f_N\}$, the function which maximizes the functional (13b) is the desired one. From (26b) one obtains an upper bound for $f(\mu_0\to\mu_0)$, i.e., forward scattering for any f_N used.[7]

To illustrate the application of the variational principle we now give an example. In analogy with the comparison potential technique in the inverse scattering theory,[8] we shall choose a sequence of trial functions for $f(\mu\to\mu_0)$ which is constructed from the complete set of eigenfunctions of the transport equation with *isotropic scattering*. In this way some very elegant results are obtained. Thus, we shall expand $f(\mu\to\mu_0)$ in terms of those eigenfunctions and find that the expansion coefficients satisfy decoupled integral equations analogous to Eq. (25) involving the same kernel $S_0(\mu,\mu_0)$.

For $f(\mu\to\mu_0)=1$, we have the completeness relation,

$$\delta(\mu-\mu_0)=\frac{1}{N_{0+}}\,[\mu\phi_{0+}(\mu)\phi_{0+}(\mu_0)-\mu\phi_{0-}(\mu)\phi_{0-}(\mu_0)] + \int_{-1}^{1}\frac{d\nu}{N(\nu)}\,\mu\phi_\nu(\mu)\phi_\nu(\mu_0), \tag{27}$$

where $\phi_{0\pm}(\mu)$, $\phi_\nu(\mu)$ are Case's[1] discrete and continuum eigenfunctions, respectively, defined by

$$\phi_{0\pm}(\mu)=\pm\frac{c\nu_0}{2}\frac{1}{\pm\nu_0-\mu}, \tag{28a}$$

$$\phi_\nu(\mu)=\frac{c\nu}{2}P\frac{1}{\nu-\mu}+\lambda(\nu)\delta(\nu-\mu), \tag{28b}$$

$$\lambda(\nu)=\tfrac{1}{2}[\Lambda^+(\nu)+\Lambda^-(\nu)], \tag{28c}$$

$$\Lambda(z)=1-\frac{cz}{2}\int_{-1}^{1}\frac{d\mu}{z-\mu}, \tag{28d}$$

$$\Lambda(\nu_0)=0, \tag{28e}$$

$$N_{0\pm}=\pm\frac{c\nu_0^3}{2}\left(\frac{c}{\nu_0^2-1}-\frac{1}{\nu_0^2}\right), \tag{28f}$$

$$N(\nu)=\Lambda^+(\nu)\Lambda^-(\nu). \tag{28g}$$

These eigenfunctions are orthogonal so that

$$\int_{-1}^{1} d\mu\, \phi_{0+}(\mu)\phi_{0-}(\mu)=0, \tag{29a}$$

$$\int_{-1}^{1} d\mu\mu\, \phi_{0\pm}^2(\mu)=N_{0\pm}, \tag{29b}$$

$$\int_{-1}^{1} d\mu\mu\, \phi_\nu(\mu)\phi_{\nu'}(\mu)=N(\mu)\delta(\nu-\nu'). \tag{29c}$$

We remark here that this set of eigenfunctions seems to be rather natural for the expansion of $f(\mu\to\mu_0)$ for the reason that, as we shall see later, for both extreme cases $f(\mu\to\mu_0)=1$ (isotropic) and $f(\mu\to\mu_0)=2\delta(\mu-\mu_0)$, the variational principle gives exact results. Now consider the expansion,

$$f(\mu\to\mu_0)=\frac{1}{N_{0+}}[\phi_{0+}(\mu_0)B_{0+}(\mu)-\phi_{0-}(\mu)B_{0-}(\mu)]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu_0)B_\nu(\mu) \qquad (30)$$

so that

$$B_{0\pm}(\mu_0)=\int_{-1}^{1}d\mu\mu f(\mu\to\mu_0)\phi_{0\pm}(\mu) \qquad (31a)$$

and

$$B_\nu(\mu)=\int_{-1}^{1}d\mu\mu f(\mu\to\mu_0)\phi_\nu(\mu). \qquad (31b)$$

Since $f(\mu\to\mu_0)$ is symmetric, the expansion (30) on the right-hand side must also be symmetric in μ,μ_0. Inserting Eq. (30) in the integral equation (25) we get

$$\frac{1}{N_{0+}}[\phi_{0+}(\mu)B_{0+}(\mu_0)-\phi_{0-}(\mu)B_{0-}(\mu_0)]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu)B_\nu(\mu_0)$$
$$=S_0(\mu,\mu_0)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu_0)\times\left\{\frac{1}{N_{0+}}[\phi_{0+}(\mu)B_{0+}(\mu')-\phi_{0-}(\mu)B_{0-}(\mu')]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu)B_\nu(\mu')\right\}. \qquad (32)$$

Upon using the orthogonality properties (29a, b, c) in Eq. (32), we obtain

$$B_{0\pm}=A_{0\pm}(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)B_{0\pm}(\mu') \qquad (33a)$$

and

$$B_\nu(\mu)=A_\nu(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)B_\nu(\mu'), \qquad (33b)$$

where

$$A_{0\pm}(\mu)=\int_{-1}^{1}d\mu'\mu'\,\phi_{0\pm}(\mu')S_0(\mu',\mu) \qquad (34a)$$

$$A_\nu(\mu)=\int_{-1}^{1}d\mu'\mu'\,\phi_\nu(\mu')S_0(\mu',\mu). \qquad (34b)$$

By comparing integral equations (33a, b) with Eq. (25) we see the obvious parallel and note that variational principles analogous to that for Eq. (25) can be set up for $B_{0\pm}(\mu)$ and $B_\nu(\mu)$ to solve the inverse problem. As mentioned earlier the advantage of dealing with the integral equations for expansion coefficients $B_{0\pm}$ and B_ν is that in both extreme cases of isotropic scattering $[f(\mu\to\mu_0)=1]$ and when $f(\mu\to\mu_0)$ is purely forward peaked [monodirectional, i.e., $\frac{1}{2}f(\mu\to\mu_0)=\delta(\mu-\mu_0)$] the variational principle provides exact solutions for $f(\mu\to\mu_0)$. We may see that as follows: Consider Eq. (33a) for $B_{0\pm}(\mu_0)$ and let the trial function be

$$\Phi_{0\pm}(\mu)=\alpha_{0\pm}A_{0\pm}(\mu) \qquad (35)$$

where $\alpha_{0\pm}$ is a discrete parameter. Also consider the functional for (33a) analogous to (13b)

$$F_{0\pm}[\Phi_{0\pm}]=\int_{-1}^{1}d\mu\,\Phi_{0\pm}(\mu)\left[2A_{0\pm}(\mu)-\Phi_{0\pm}(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)\Phi_{0\pm}(\mu')\right]. \qquad (36)$$

Put $\Phi_{0\pm}(\mu)=\alpha_{0\pm}A_{0\pm}(\mu)$ in (36) to obtain

$$F_{0\pm}[A_{0\pm}]=\alpha_{0\pm}(q-\alpha_{0\pm})P_{0\pm}-\frac{c}{2}\alpha_{0\pm}^2Q_{0\pm}, \qquad (37)$$

where

$$P_{0\pm}=\int_{-1}^{1}d\mu\,A_{0\pm}^2(\mu), \qquad (38a)$$

$$Q_{0\pm}=\int_{-1}^{1}d\mu\,A_{0\pm}(\mu)\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)A_{0\pm}(\mu'). \qquad (38b)$$

Maximization of the functional $F_{0\pm}[A_{0\pm}]$, defined by Eq. (37), yields

$$\alpha_{0\pm}=\left(1+\frac{c}{2}\frac{Q_{0\pm}}{P_{0\pm}}\right)^{-1}. \qquad (39)$$

Similarly for the continuum expansion coefficients $B_\nu(\mu)$ of Eq. (33b), if we consider the trial function

$$\Phi_\nu(\mu)=\alpha_\nu A_\nu(\mu), \qquad (40)$$

then the maximization of the functional

$$F_\nu[\Phi_\nu]=\int_{-1}^{1}d\mu\,\Phi_\nu(\mu)\left[2A_\nu(\mu)-\Phi_\nu(\mu)-\frac{c}{2}\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)\Phi_\nu(\mu')\right] \qquad (41)$$

yields

$$\alpha_\nu=\left(1+\frac{c}{2}\frac{Q_\nu}{P_\nu}\right)^{-1}, \qquad (42)$$

where

$$P_\nu=\int_{-1}^{1}d\mu\,A_\nu^2(\mu), \qquad (43a)$$

$$Q_\nu=\int_{-1}^{1}d\mu\,A_\nu(\mu)\int_{-1}^{1}d\mu'\,S_0(\mu',\mu)A_\nu(\mu'). \qquad (43b)$$

Hence, by replacing $B_{0\pm}$ by $\Phi_{0\pm}$ and B_ν by Φ_ν in Eq. (30), we have an approximate representation of the phase function, which is given by

$$f(\mu\to\mu_0)\cong\frac{1}{N_{0+}}\left[\phi_{0+}(\mu_0)A_{0+}(\mu)\Big/\left(1+\frac{c}{2}\frac{Q_{0+}}{P_{0+}}\right)-\phi_{0-}(\mu_0)A_{0-}(\mu)\Big/\left(1+\frac{c}{2}\frac{Q_{0-}}{P_{0-}}\right)\right]+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\phi_\nu(\mu_0)A_\nu(\mu)\Big/\left(1+\frac{c}{2}\frac{Q_\nu}{P_\nu}\right). \qquad (44)$$

We may now check the accuracy of the representation (44) for the two extreme cases when

$$f(\mu\to\mu_0)=\begin{cases}1 & \text{isotropic}\\ 2\delta(\mu-\mu_0) & \text{monodirectional.}\end{cases} \qquad (45)$$

For the two cases we conclude from Eq. (25) that

$$S_0(\mu,\mu_0)=\begin{cases}\dfrac{1}{1-c} & \text{isotropic,}\\ \dfrac{2}{1-c}\,\delta(\mu-\mu_0) & \text{monodirectional.}\end{cases} \tag{46}$$

From Eqs. (38a, b) and Eqs. (43a, b) we obtain

$$\frac{Q_{0\pm}}{P_{0\pm}}=\frac{Q_\nu}{P_\nu}=\frac{2}{1-c} \tag{47}$$

for *both* isotropic and monodirectional phase functions, while

$$\left.\begin{aligned}A_{0\pm}&=\pm\nu_0\\ A_\nu(\mu)&=\nu\end{aligned}\right\}\quad\text{isotropic} \tag{48}$$

and

$$\left.\begin{aligned}A_{0\pm}(\mu)&=\frac{2}{(1-c)}\,\mu\phi_{0\pm}(\mu)\\ A_\nu(\mu)&=\frac{2}{1-c}\,\mu\phi_\nu(\mu)\end{aligned}\right\}\quad\text{monodirectional.} \tag{49}$$

Thus for the isotropic case we conclude from Eqs. (44) and (48) that

$$f(\mu\to\mu_0)=(1-c)\left(\frac{\nu_0}{N_{0+}}\,[\phi_{0+}(\mu_0)+\phi_{0-}(\mu_0)]+\int_{-1}^{1}\frac{d\nu\,\nu}{N(\nu)}\right). \tag{50}$$

But the right-hand side of Eq. (50) is merely a unity. This is easily seen by noting that Case's eigenfunctions satisfy the following relations:

$$\int_{-1}^{1}d\mu\mu\,\phi_\nu=\nu(1-c), \tag{51a}$$

$$\int_{-1}^{1}d\mu\mu\,\phi_{0\pm}(\mu)=(\pm\nu_0)(1-c). \tag{51b}$$

An integration of the completeness relation (27) with respect to μ gives

$$(1-c)\left(\frac{\nu_0}{N_{0+}}\,\phi_{0+}(\mu_0)+\phi_{0-}(\mu_0)+\int_{-1}^{1}\frac{d\nu\,\nu}{N(\nu)}\right)=1. \tag{52}$$

Hence for the isotropic case we obtain in Eq. (50), $f(\mu\to\mu_0)=1$ (as expected). For the monodirectional phase function, substitution of Eqs. (49) and (47) in Eq. (44) merely reproduces the completeness relation (27) save for a factor of two so that $f(\mu\to\mu_0)=2\delta(\mu-\mu_0)$; the factor of two should be there because of the normalization so that

$$\tfrac{1}{2}\int_{-1}^{1}d\mu\,f(\mu\to\mu_0)=1.$$

In conclusion we wish to point out that for the *direct problem* one can actually calculate the coefficients $A_{0\pm}(\mu)$ and $A_\nu(\mu)$ in the approximate eigenfunction representation (44) of $f(\mu\to\mu_0)$. Let us assume that the right-hand side of Eq. (44) is an exact representation for some function $f_0(\mu\to\mu_0)$ so that

$$f_0(\mu\to\mu_0)=\frac{1}{N_{0+}}\,(\phi_{0+}(\mu_0)A_{0+}(\mu)\alpha_{0+}-\phi_{0-}(\mu_0)A_{0-}(\mu)\alpha_{0-})+\int_{-1}^{1}\frac{d\nu}{N(\nu)}\,\phi_\nu(\mu_0)A_\nu(\mu)\alpha_\nu \tag{53}$$

where $\alpha_{0\pm}$, α_ν are defined by (39) and (42) respectively. This function then corresponds to a situation in which some (abstract) operator is defined to have the spectral density

$$\begin{aligned}\frac{d\rho_0(\nu)}{\nu}&=\frac{d\nu}{N(\nu)\alpha_\nu}\qquad -1\leqslant\nu\leqslant 1\\ &=\frac{\alpha_{0+}\delta(\nu-\nu_{0+})d\nu}{N_{0+}}-\frac{\alpha_{0-}\delta(\nu-\nu_{0-})d\nu}{N_{0+}}\qquad |\nu|>1.\end{aligned} \tag{54}$$

Thus, one may use $f_0(\mu\to\mu_0)$, in analogy with inverse scattering, as a sort of higher order comparison potential. This offers an alternate method of dealing with the direct problem when the Legendre expansion is not desirable in the main transport equation when $f(\mu\to\mu_0)$ is highly anisotropic.

For the purpose of the direct problem we now calculate the coefficients $A_{0\pm}(\mu)$ and $A_\nu(\mu)$. Thereby we obtain a relation between the basis used by Case[2] and the Legendre polynomials

$$A_\nu(\mu)=\int_{-1}^{1}d\mu'\mu'\,\phi_\nu(\mu')S_0(\mu',\mu). \tag{34b}$$

From Eq. (16) we have

$$S_0(\mu',\mu)=\sum_{n=0}^{\infty}\frac{2n+1}{2}\,P_n(\mu')q_n(\mu). \tag{55}$$

Solving Eq. (18) for $q_n(\mu)$ and substituting that in (55) yields

$$S_0(\mu',\mu)=\sum_{n=0}^{\infty}(2n+1)\,\frac{f_n}{1-cf_n}\,P_n(\mu')P_n(\mu). \tag{56}$$

From (34b) and (56), we get

$$A_\nu(\mu)=\sum_{n=0}^{\infty}(2n+1)\,\frac{f_n}{1-cf_n}\,P_n(\mu)\nu(W_n(\nu)-c\delta_{n0}, \tag{57}$$

where

$$W_n(\nu)=\int_{-1}^{1}d\mu\,\phi_\nu(\mu)P_n(\mu) \tag{58}$$

and used the fact that

$$\mu\phi_\nu(\mu)=\nu\phi_\nu(\mu)-\nu c/2. \tag{59}$$

Actually, $W_n(\nu)$ are a special case of the orthogonal polynomials obtained from the transport equation by direct expansion of $f(\mu'\to\mu)$ in terms of Legendre polynomials. In other words, when $f_n=\delta_{n0}$, we have from (58), (59) and the recurrence relation

$$(2n+1)\mu P_n(\mu)=(n+1)P_{n+1}(\mu)+nP_{n-1}(\mu) \tag{60}$$

the three-term pure recurrence relation for $W_n(\nu)$:

$$(n+1)W_{n+1}(\nu)+nW_{n-1}(\nu)=(2n+1)(1-c\delta_{n0})\nu W_n(\nu),\quad n\geqslant 0 \tag{61}$$

with $W_0(\nu)=1$ and $W_{-1}(\nu)=0$. That Eq. (61) is a pure three-term recurrence relation implies that $W_n(\nu)$ are orthogonal and vice versa. But, more importantly, $W_n(\nu)$ can be easily calculated either from Eq. (61) or from its explicit representation (58). All these considerations apply for $A_{0\pm}(\mu)$ by merely replacing ν by $\pm\nu_0$, where ν_0 is the zero of the dispersion function (28e). We shall defer any further discussion of the direct problem until Part 2, which will be a sequel to this paper.

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Corrigendum to "Covariant inverse problem of the calculus of variations"

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Our recent work showing how to obtain covariant Lagrangians for certain classes of field equations is extended to cases where the field appears to order λ^m ($m \leq -2$) and also to cases where metric variations are taken.

In a recent paper[1] we demonstrated that the requirement that the Frechet derivative, $F^{|B}\phi_B$, of the expression

$$F=\int y_A\int_0^1 N^A(\lambda y_{B,\pi})(-g)^{1/2}\,d\lambda\,d^4x, \tag{1}$$

vanishes, implies the Euler–Lagrange equations

$$N^A(y_B)=0, \tag{2}$$

provided $N^A(y)$ satisfies the conditions

$$\frac{\partial N^A}{\partial y_{B,\pi}}=\sum_\Lambda(-1)^{|\pi+\Lambda|}\binom{\pi+\Lambda}{\Lambda}\times(-g)^{-1/2}\left((-g)^{-1/2}\frac{\partial N^B}{\partial y_{A,\pi\Lambda}}\right)_{,\Lambda}. \tag{3}$$

Following Atherton and Homsy[2] we have called such differential operators potential operators.

It has been brought to our attention[3] that there exist certain operators, primarily of mathematical interest, for which the expression (1) is of order λ^m, $m\leqslant -1$ and consequently the integral diverges.[4] It is our purpose here to show how the Lagrangians may be obtained for $m\leqslant -2$.[5] We shall consider two cases. First, those in which fields other than the metric are varied; second, those in which the metric is varied.

Consider a potential operator $N^A(y_{B,\pi})$ for which one or more terms of $N^A(y_{B,\pi})$ are of order $m_1, m_2, \cdots \leqslant -2$ in λ. Referring to the potential conditions, (3), it may be seen that terms of differing order m_i in λ must be individually potential. Thus, for simplicity, and without loss of generality, we may consider $N^A(y_{B,\pi})$ to consist only of terms of the same order.

Replace Eq. (1) by the more general expression

$$F=\int y_A\int_0^1 f(\lambda)N^A(\lambda y_B)(-g)^{1/2}\,d\lambda\,d^4x. \tag{4}$$

[We note that if $N^A(\lambda y_{B,\pi})$ *is entirely of order* $m\geqslant 0$ in λ, $f(\lambda)\equiv 1$ and the result in Ref. 1 holds.] If we set $f(\lambda)=c_m\lambda^m$ and proceed to compute the Frechet derivative of (4), $F^{|B}\phi_B$, as in Ref. (1), we obtain

$$F^{|B}\phi_B=\int \phi_A c_m(1-m)N^A(y_{B,\pi})(-g)^{1/2}\,d^4x, \tag{5}$$

Setting $c_m=(1-m)^{-1}$, we find that the requirement that $F^{|B}\phi_B=0$ implies the Euler–Lagrange equations

$$N^A(y_{B,\pi})=0. \tag{6}$$

If $N^A(\lambda y_{B,\pi})$ is homogeneous in λ,

$$F=\int (1-m)^{-1}y_A N^A(y_{B,\pi})(-g)^{1/2}\,d^4x. \tag{7}$$

When metric variations are considered, questions of convergence of the λ integral in Eq. (1) may be obviated by the appropriate choice of independent variants. We prefer to consider $g^{\mu\nu}$ as the varied field with covariant (rather than contravariant) source terms. This has the merit of making the order of all terms in an expression such as

$$N_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R-kT_{\mu\nu}(E)=0. \tag{8}$$

positive. Here $T_{\mu\nu}(E)$ is the electromagnetic stress-energy tensor $F_{\mu\rho}F_\nu{}^\rho-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$, with $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$. Note that if one took $g_{\mu\nu}$ and A_μ as independent variants instead, the order of $T_{\mu\nu}(E)$ would be λ^{-1} and our formalism would not be applicable.

We replace Eq. (1) in these situations by the equally acceptable alternative

$$F=\int y^A\int_0^1 N_A(\lambda y^B{}_{,\pi})d\lambda(-g)^{1/2}\,d^4x. \tag{9}$$

It is instructive to obtain the equations of motion for an operator $N_{\mu\nu}(g^{\rho\sigma}{}_{,\pi})$ from (9) by requiring that the Frechet derivative, $F_{|B}\phi^B$, vanish. We assume at the outset that $N_{\mu\nu}$ is potential. The Frechet derivative of (9) is

$$F_{|B}\phi^B=\int \phi^{\mu\nu}\left\{\int_0^1 N_{\mu\nu}(\lambda g^{\alpha\beta})d\lambda +g^{\mu\nu}\int_0^1 N_{\mu\nu|\lambda\rho\sigma}\phi^{\rho\sigma}\,d\lambda -g^{\rho\sigma}\int_0^1 N_{\rho\sigma}(\lambda g^{\alpha\beta})\phi^{\mu\nu}g_{\mu\nu}\,d\lambda\right\}(-g)^{1/2}\,d^4x. \tag{10}$$

The last term in (10) arises from the presence of $(-g)^{1/2}$ in Eq. (9). Following the procedure in Ref. 1 we obtain, after several steps,

$$F_{|B}\phi^B=\int \phi^{\mu\nu}\left\{N_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\int_0^1 N_{\rho\sigma}(\lambda g^{\alpha\beta})\right\}\times(-g)^{1/2}\,d^4x. \tag{11}$$

Requiring that $F_{|B}\phi^B=0$, we obtain for the equations of motion

$$N_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\int_0^1 N_{\rho\sigma}(\lambda g^{\alpha\beta})d\lambda=0. \tag{12}$$

$N_{\sigma\rho}(\lambda g^{\alpha\beta})$ in the second term of Eq. (12) will, in general, consist of a sum of source terms of various orders, m, in λ which we may indicate by writing

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$$N_{\rho\sigma}(\lambda g^{\alpha\beta})=\sum_{m>0}\lambda^m N_{(m)\rho\sigma}\,.$$

We carry out the integration in Eq. (12) and obtain

$$N_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\sum_m(1+m)^{-1}g^{\rho\sigma}N_{(m)\rho\sigma}=0. \qquad (13)$$

For every order of λ^m we may rewrite $N_{(m)\mu\nu}$ in terms of its trace free part $\bar{N}_{(m)\mu\nu}$ and its trace to transform Eq. (13) into

$$\bar{N}_{(m)\mu\nu}+(1/n)g_{\mu\nu}N_{(m)}-\tfrac{1}{2}g_{\mu\nu}(m+1)^{-1}N_{(m)}=0.$$

where $N_{(m)}=g^{ps}N_{(m)ps}$. This is equivalent to

$$\bar{N}_{\mu\nu}-g_{\mu\nu}[n-2(m+1)][2n(m+1)]^{-1}N_{(m)}=0.$$

Thus, we obtain from Eq. (9) trace-free equations of motion whenever $n-2(m+1)=0$. The stress-energy tensor for the electromagnetic field in four dimensions is the simplest example of a trace free term of this type (with $m=1$).

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ERRATA

Erratum: The electromagnetic field on a simplicial net [J. Math. Phys. 16, 2432 (1975)]

Rafael Sorkin

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(Received 1 March 1978)

P. 2432: A "1" and an "*n*" have been transposed in Eq. (5), which should read

$$\langle \mathbf{e}^j,\mathbf{e}_k\rangle=\tilde{\delta}^j_k\equiv\delta^j_k-\frac{1}{n+1}=\begin{cases}\dfrac{n}{n+1} & \text{if } j=k,\\[2mm] \dfrac{-1}{n+1} & \text{if } j\neq k.\end{cases}$$

P. 2433 (line 14): In place of "... its *affine components* $T^{j\cdots k}_{i\cdots m}$" read "... its *affine components* $\tilde{T}^{j\cdots k}_{i\cdots m}$."

The same change should be made in Eq. (9) and Eq. (10) [but the "T" on the lhs of (9) should be left as it is].

P. 2435 (line 19): In place of "... since $e_{k_0}\,e_{k_0}=0$, and ..." read "... since $e_{k_0}\wedge e_{k_0}=0$, and ...".

$$N_{\rho\sigma}(\lambda g^{\alpha\beta}) = \sum_{m>0} \lambda^m N_{(m)\rho\sigma}.$$

We carry out the integration in Eq. (12) and obtain

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